LECTURE 19

(Friday NOV. 15, 2019)

– Recall:

<u>Thm</u> S_n is generated by transpositions.
(I.e., every $\alpha \in S_n$ can be written

$$\alpha = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_N$$

where the α_i are transpositions.)
↖ usually <u>not</u> disjoint.

VERY non-unique.
(ex. $e = (ab)(ab)$.)
However, the <u>parity</u> of N turns out to be completely determined by α.

$$S_n = \underbrace{\{\underline{\text{even}} \text{ perm.}\}}_{A_n} \cup \underbrace{\{\underline{\text{odd}} \text{ perm.}\}}_{(12)A_n}$$

— not a subgroup.

Here $|A_n| = \frac{n!}{2}$.

<u>Remark</u>: S_n is generated by <u>simple</u> transpositions $(a, a+1)$
↖ neighbor swaps.

<u>Idea</u>: When $a+1 < b$,

$$(ab) = (a, a+1)(a+1, b)(a, a+1)$$

Keep using this formula.

∘ <u>Why</u>? RHS takes

$a \to a+1 \to b \to b$

$a+1 \to a \to a \to a+1$

$b \to b \to a+1 \to a$.

✓

Ex $(25) = (23)(35)(23) = (23)(34)(45)(34)(23)$.

$(n \geq 2)$

Thm. There is a unique non-trivial homomorphism
$$f : S_n \longrightarrow \{\pm 1\}.$$
It takes any transposition to -1.

$N \pmod 2$ dep. only on α.

PROOF. — in particular $f(\alpha) = f(\alpha_1) \cdots f(\alpha_N) = (-1)^N$.

Any such f must sat. $f(a,b) = -1$. Otherwise, if $f(a,b) = +1$ for some (ab), then $f(c,d) = +1$ for all (cd). (Write $(cd) = \sigma (ab) \sigma^{-1}$.) Follows that $f(\alpha) = +1$, $\forall \alpha \in S_n$.

(i.e., f trivial).

→ This also shows the uniqueness of f:

If f_1 and f_2 agree on a set of generators they agree on all $\alpha \in S_n$.

(cf. linear transformations / bases)

EXISTENCE? Introduce the polynomial
$$P(x_1, \ldots, x_n) = \prod_{i<j} (x_i - x_j)$$
(n variables, coefficients in $\mathbb{Z}$.)

Ex $(n=3)$ $P(x_1, x_2, x_3) = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$.

Notation: $\alpha \in S_n$ permutes the variables,

$$P^{\alpha}(x_1, \dots, x_n) := P(x_{\alpha(1)}, \dots, x_{\alpha(n)}) .$$

Ex ($n=3$) $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$.

$$P^{\alpha}(x_1, x_2, x_3) = P(x_2, x_3, x_1) =$$

$$(x_2 - x_3)(x_2 - x_1)(x_3 - x_1) = P(x_1, x_2, x_3)$$

$\cdot(-1)$ $\cdot(-1)$

$\beta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$.

$$P^{\beta}(x_1, x_2, x_3) = P(x_2, x_1, x_3) =$$

$$(x_2 - x_1)(x_2 - x_3)(x_1 - x_3) = - P(x_1, x_2, x_3).$$

$\cdot(-1)$

— in general: For $\alpha \in S_n$,

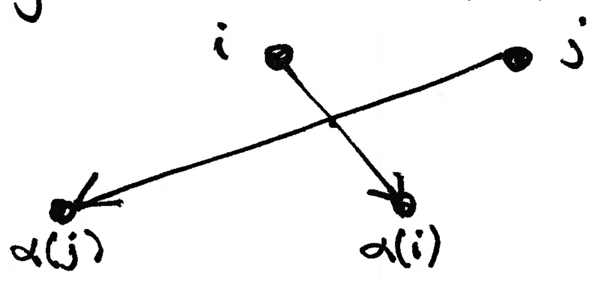
$$P^{\alpha}(x_1, \dots, x_n) = \pm P(x_1, \dots, x_n)$$

sign is $(-1)^{c(\alpha)}$,

where

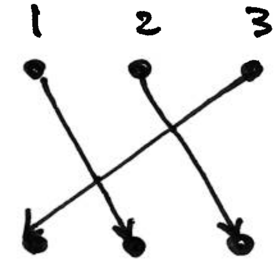
$$c(\alpha) = \#\{(i,j) \mid i<j \text{ and } \alpha(i) > \alpha(j)\}$$

(number of "crossings")



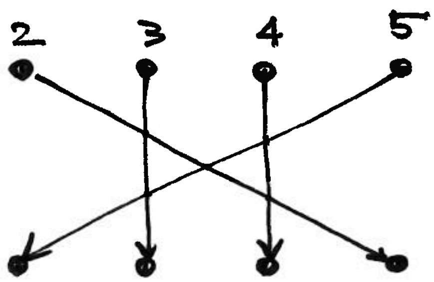
EX ($n=3$) cont. $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ has $c(\alpha) = 2$

$\beta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ has $c(\beta) = 1$



[Note: Any simple transposition $\alpha = (a, a+1)$ has $c(\alpha) = 1$]

EXC (25) has $c = 5$:



— the KEY property: $\forall \alpha, \beta \in S_n$,

$$p^{\alpha\beta}(x_1, \dots, x_n) = (p^{\alpha})^{\beta}(x_1, \dots, x_n)$$

(indeed RHS $= p^{\alpha}(x_{\beta(1)}, \dots, x_{\beta(n)})$

$$= p(x_{\alpha\beta(1)}, \dots, x_{\alpha\beta(n)}) \;.\quad)$$

Now,

$$p^{\alpha\beta} = (-1)^{c(\alpha\beta)} \cdot p$$

— suppress variables $x_1, \dots, x_n$.

$$\|$$

$$(p^{\alpha})^{\beta} = (-1)^{c(\beta)} \cdot p^{\alpha} = (-1)^{c(\beta) + c(\alpha)} p .$$

rename the variables $y_i = x_{\beta(i)}$.

Shows: $c(\alpha\beta) \equiv c(\alpha) + c(\beta) \pmod 2$.

— in other words:

$$f(\alpha) := (-1)^{c(\alpha)}$$ defines a homomorphism $f: S_n \longrightarrow \{\pm 1\}$.

Non-trivial: $f(\alpha) = -1$

α ← (simple) transposition. $\square$

Def. $$\operatorname{sign}(\alpha) = \begin{cases} +1, & N \text{ even} \\ -1, & N \text{ odd} \end{cases}$$ (well-def.)

$$\operatorname{sign}(\alpha\beta) = \operatorname{sign}(\alpha)\operatorname{sign}(\beta).$$

Recall,

$$\operatorname{sign}(m\text{-cycle}) = (-1)^{m-1}.$$

using

$$(x_1 x_2 \dots x_m) = \underbrace{(x_1 x_m)(x_1 x_{m-1}) \cdots (x_1 x_2)}_{m-1 \text{ transpositions}}.$$