LECTURE 19
(Friday Nov. 15, 2019)
- Recall:

Theorem: $S_n$ is generated by transpositions.
(I.e., every $\alpha \in S_n$ can be written
$$\alpha = \alpha_1 \circ \alpha_2 \circ \ldots \circ \alpha_N$$
where the $\alpha_i$ are transpositions.)

VERY non-unique. $\Gamma$ usually not disjoint.
(ex. $e = (ab)(ab)$.)

However, the parity of $N$ turns out to be completely determined by $\alpha$.

$$S_n = \{\text{even perm.}\} \cup \{\text{odd perm.}\}$$

$An \quad (12)an \quad \text{not a subgroup.}$

Here, $|An| = \frac{n!}{2}$.

Remark: $S_n$ is generated by simple transpositions $(a, a+1)$

Idea: When $a+1 < b$,

$$(ab) = (a, a+1)(a+1, b)(a, a+1)$$

keep using this formula.

\[\begin{align*}
\text{Why? RHS takes} & \\
\text{a} & \rightarrow a+1 \rightarrow b \rightarrow b \\
a+1 & \rightarrow a \rightarrow a \rightarrow a+1 \\
b & \rightarrow b \rightarrow a+1 \\
\end{align*}\]
\[ \textbf{Thm.} \text{ There is a unique non-trivial homomorphism } \psi \colon S_n \to \{ \pm 1 \}. \]

\[ \psi(\alpha) = \psi(\alpha_1) \cdots \psi(\alpha_N) = (-1)^N. \]

\[ \text{Any such } \psi \text{ must sat. } \psi(\alpha, \beta) = -1. \text{ Otherwise, if } \psi(\alpha, \beta) = +1 \text{ for some } (\alpha, \beta), \text{ then } \psi(\gamma, \delta) = +1 \text{ for all } (\gamma, \delta). \]

\[ \text{(Write } (\gamma, \delta) = \sigma(\alpha, \beta)\sigma^{-1}) \text{ follows that } \psi(\alpha) = +1, \forall \alpha \in S_n. \text{ (i.e., } \psi \text{ trivial).} \]

\[ \text{This also shows the uniqueness of } \psi: \]

\[ \text{If } \psi_1 \text{ and } \psi_2 \text{ agree on a set of generators } \alpha, \beta, \text{ they agree on all } \alpha \in S_n. \]

\[ \text{(cf. linear transformations/bases)} \]

**EXISTENCE?**

\[ \text{Introduce the polynomial } \]

\[ P(x_1, \ldots, x_n) = \prod_{i<j} (x_i - x_j) \]

\[ \text{(n variables, coefficients in } \mathbb{Z} \text{)} \]

Ex(n=3) \[ P(x_1, x_2, x_3) = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3). \]
Notation: \( \alpha \in S_n \) permutes the variables,

\[
P^\alpha(x_1, \ldots, x_n) := P(x_{\alpha(1)}, \ldots, x_{\alpha(n)})
\]

\( \text{Example (n=3)} \quad \alpha = (\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 1 \end{array}) \).

\[
P^\alpha(x_1, x_2, x_3) = P(x_2, x_3, x_1) = 
(x_2 - x_3)(x_2 - x_1)(x_3 - x_1) = P(x_1, x_2, x_3)
\]

\( \beta = (\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 1 & 3 \end{array}) \).

\[
P^\beta(x_1, x_2, x_3) = P(x_2, x_1, x_3) = 
(x_2 - x_1)(x_2 - x_3)(x_1 - x_3) = - P(x_1, x_2, x_3)
\]

\[\text{(-1)} \text{ \hspace{1cm} (-1)}\]

--- in general: For \( \alpha \in S_n \),

\[
P^\alpha(x_1, \ldots, x_n) = \pm P(x_1, \ldots, x_n)
\]

\[
\text{sign } \alpha = (-1)^c(\alpha),
\]

where \( c(\alpha) = \# \{ i < j \text{ and } \alpha(i) > \alpha(j) \} \)

(number of crossings =)

\[
\text{example:} \quad \alpha(1) = 2, \quad \alpha(2) = 1, \quad \alpha(3) = 3
\]

\[
\alpha(1) \quad \alpha(2) \quad \alpha(3)
\]

\[
\alpha(i) \quad \alpha(j)
\]
\( \text{Ex}(n=3) \) cont. \( \alpha = (1 \ 2 \ 3) \) has \( c(\alpha) = 2. \) 
\( \beta = (2 \ 3 \ 1) \) has \( c(\beta) = 1. \)

[Note: Any simple transposition \( \alpha = (a, a+1) \) has \( c(\alpha) = 1. \)]

Exc \( (25) \) has \( c = 5. \)

The key property: \( \forall \alpha, \beta \in S_n, \)

\[
p^{\alpha \beta}(x_1, \ldots, x_n) = (p^\alpha)^\beta(x_1, \ldots, x_n)
\]

(Indeed \( \text{RHS} = p^\alpha(x_{\beta(1)}, \ldots, x_{\beta(n)}) \))

= \( p(x_{\alpha \beta(1)}, \ldots, x_{\alpha \beta(n)}) \).

Now,

\[
p^{\alpha \beta} = (-1)^{c(\alpha \beta)} \cdot p
\]

\[
(p^\alpha)^\beta = (-1)^{c(\beta)} \cdot \alpha = (-1)^{c(\beta)+c(\alpha)} \cdot p.
\]

\( \uparrow \text{rename the variables } y_i = x_{\beta(i)}. \)
Shown: \( c(\alpha \beta) \equiv c(\alpha) + c(\beta) \pmod{2} \).

--- in other words:

\[
f(\alpha) := (-1)^{c(\alpha)} \quad \text{defines a homomorphism}
\]

\[
f: S_n \rightarrow \{ \pm 1 \}.
\]

Non-trivial: \( f(\alpha) = -1 \)

\( \text{(simple) transposition} \). \( \square \)

**Def.** \( \text{sign}(\alpha) = \begin{cases} +1 & , \ N \text{ even} \\ -1 & , \ N \text{ odd} \end{cases} \) (well-def.)

\[
\text{sign}(\alpha \beta) = \text{sign}(\alpha) \cdot \text{sign}(\beta).
\]

Recall,

\[
\text{sign(m-cycle)} = (-1)^{m-1}.
\]

Using

\[
(x_1 x_2 \ldots x_m) = (x_1 x_m)(x_1 x_{m-1}) \cdots (x_1 x_2)
\]

\( m-1 \) transpositions.