LECTURE 22
(Friday Nov. 22, 2019)
\((G, \ast)\) group with subgroup \(H \leq G\).

- **Cosets** are subsets \(a \ast H = \{a \ast h : h \in H\} = [a]\).

- **Partition** \(G\) (i.e., every \(a \in G\) lies in exactly one coset).

- Collection of all cosets:
  \[ a \ast H = b \ast H \iff a^{-1}b \in H. \]

- \(G/H = \{a \ast H : a \in G\}\).

  - **Set**.  **Index** = \([G:H] = |G/H|\).
  - (Lagrange: \(|G| = |G:H| \cdot |H|\))

  - Does \(G/H\) have a **group structure**?
    - (Similar to \(+ \) on \(\mathbb{Z}_N\) given by \([a] + [b] = [a+b]\)).
    - **TRY:** Define a **composition law** \(\bullet\) on \(G/H\) by
      \[(a \ast H) \bullet (b \ast H) = (a \ast b) \ast H.\]
      - more briefly: \([a] \bullet [b] = [a \ast b]\).

  - **Problem:** Not always well-defined! May happen that \([a] = [a']\) and \([b] = [b']\) but \([a \ast b] \neq [a' \ast b']\).
rot. by $2\pi/N$  \[ \text{ref.} \]

\[ EX \ D_N = \langle y, s \rangle, \quad r^N = s^2 = e \quad \text{and} \quad rs = sr^{-1}. \]

(Symmetry of an $N$-gon)  \[ \text{Let} \ H = \langle s \rangle = \{ e, s, s^2 \}. \]

Note:  \[ rH = rsH \quad \text{(some} \ s \in H) \quad \text{but} \]

\[ r^2H \neq H \quad \text{since} \quad r^2 \notin H \quad \text{(assuming} \ N \geq 3) \]

\[ (rs)^2H = rsrsH = srsr^{-1}rsH = s^2H = H \]

So \[ [r] = [rs] \quad \text{but} \quad [r*r] \neq [rs*rs] \]

--- shows \[ \text{not well-def. composition law on} \ D_N/H. \]

Def.  \[ H \leq G \] is a \underline{normal} subgroup if

\[ a \ast H = H \ast a \quad \forall a \in G. \]

Remark: This does \underline{not} mean \[ a \ast h = h \ast a \] for all \[ h \in H. \] It means: For any two \[ a \in G, h \in H \]

there's an \[ h' \in H \] with \[ a \ast h = h' \ast a \] — and

vice versa. Possibly \[ h' \neq h. \]

Notation: When \[ H \leq G \] is normal we write \[ \boxed{H \triangleleft G}. \]

Obs: If \[ G \] abelian every \[ H \leq G \] is \underline{normal}. \]
Proposition: When $H$ is normal in $G$, 
- is a well-defined composition law on $G/H$.

**Proof:** Let's suppress $\ast$ in this proof (i.e. write $aH = a \ast H$ etc.). Assume $aH = bH \land bH = b'H$. We must show $abH = a'b'H$. Now,

$$abH = a'b'H = ab'H = aHb' = a'Hb' = a'b'H$$

Thus $(G/H, \cdot)$ is a group when $H \triangleleft G$:
- **Associative**, neutral element $e \ast H = H$
- **Inverse** of $a \ast H$ is $a^{-1} \ast H$.

"**Quotient Group** of $G$ by $H$:"

EX (cont) $H = \{e, s^2\}$ not normal in $D_N$:

$$rH = \{r, rs\} \neq \{r, rs\}$$

$Hr = \{r, sr\}$

and $rs = s^r \neq sr$ since $N > 2$. Compare with $(\mathbb{Z}_N, +)$.
Exc. Any $H \leq G$ of index $[G:H] = 2$ is normal. 
(Hint: For $a \not\in H$, $G = H \cup aH = H \cup Ha$ so $aH = Ha = \text{complement of } H \text{ in } G$.)

Ex.: $A_n \triangleleft S_n$ and $\langle r \rangle \triangleleft D_n$.

Lemma ("Normality Criterion"): Let $H \leq G$. T.F.A.E.:

1. $H$ is normal in $G$ (i.e., $aH = Ha, \forall a \in G$)
2. $aHa^{-1} = H, \forall a \in G$.

Proof. $(1) \iff (2)$: $aH = Ha$ iff $aHa^{-1} = Ha a^{-1} = H$.

$(2) \implies (3)$: Trivial.

$(3) \implies (2)$: Need the other inclusion $aHa^{-1} \supseteq H$.

- follows from $(3)$ applied to $a^{-1}$ instead of $a$. \qed
Ex. \( A_4 \) with subgroup \( V = \{ e, (12)(34), (13)(24), (14)(23) \} \).

- non-cyclic of size 4.

\( V \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \)

"Klein's 4-group" (to see this decompose \( \alpha \) into cycles?)

\( V \) is normal in \( A_4 \):

\[ S V S^{-1} \subseteq V \quad \forall S \in A_4. \]

- indeed \( \alpha^2 = e \) implies

\[ (S \alpha S^{-1})^2 = S \alpha^2 S^{-1} = e. \]

\( A_4 / V \) is a group of size \( [A_4 : V] = \frac{12}{4} = 3 \).

\( \therefore A_4 / V \cong \mathbb{Z}_3 \).
An application of quotient groups:

**Theorem.** Suppose $G$ is finite and $H \triangleleft G$. Then:

$$a^{-1} \in H \text{ for all } a \in G.$$ 

(known this for $[G:H]=2$)

**Proof.** Apply (Corollary of) Lagrange to the group $G/H$: $\forall a \in G$,

$$|G/H| (aH) = eH$$

$$\Rightarrow$$

$$a [G:H] H$$