# LECTURE 24

(Wednesday NOV. 27, 2019)

*Ex*: A $m \times n$-matrix.
$f : \mathbb{R}^n \to \mathbb{R}^m$, $f(x) = Ax$
column space & null space.

*Def*

(1) The *image* of $f$ is

$$\mathrm{im}(f) = \{y \in H : y = f(x) \text{ for } \underline{\text{some}}\ x \in G\}$$
$$= \{f(x) : x \in G\}$$
$$= f(G).$$

(2) The *kernel* of $f$ is

$$\ker(f) = \{x \in G : f(x) = e\} = \overset{\substack{\text{"inverse image"} \\ \downarrow}}{f^{-1}}(\{e\}).$$

*Lemma* • $\mathrm{im}(f)$ is a *subgroup* of $H$.

• $\ker(f)$ is a *normal* *subgroup* of $G$.

PROOF. • $\mathrm{im}(f)$ *closed* under $\bullet$:

$f(a) \bullet f(b) = f(x)$ some $x \in G$, namely $x = a * b$.

$e_H \in \mathrm{im}(f)$; indeed $e_H \underset{(i)}{=} f(e_G)$. $\mathrm{im}(f)$ closed under *inversion* by (ii) above.

• $\ker(f)$ *closed* under $*$:

$$f(a) = e \wedge f(b) = e \implies f(a*b) = e \bullet e = e.$$

so $a * b \in \ker(f)$.

$e_G \in \ker(f)$ ✓ closed under $a \longmapsto a^{-1}$ ✓

(check both as an exc.)

— The key point is $\ker(f) \lhd G$ ("normality"):

Amounts to: $\forall a \in \ker(f)$ $\forall g \in G$ must check that $g * a * g^{-1} \in \ker(f)$.

$$f(g * a * g^{-1}) = f(g) \bullet \underbrace{f(a)}_{e_H} \bullet f(g)^{-1} = e_H \quad ✓ \quad \square$$

**Remark:** $\mathrm{im}(f)$ not always normal in $H$.

Ex: $f : \mathbb{Z}_3 \longrightarrow S_{\underset{4}{3}}$

$[0] \mapsto e$

$[1] \mapsto (123)$

$[2] \mapsto (132)$

has $\ker(f) = \{[0]\}$

& $\mathrm{im}(f) = \langle (123) \rangle$

↳ not normal in $S_4$:

$$(14) \circ \underset{\underset{\mathrm{im}(f)}{\Uparrow}}{(123)} \circ (14)^{-1} = (423) \notin \mathrm{im}(f).$$

**Observe:**

- $f$ surjective $\Longleftrightarrow$ $\mathrm{im}(f) = H$ ✓
- $f$ injective $\Longleftrightarrow$ $\ker(f) = \{e\}$.

(why? $\Longrightarrow$: $f(a) = e = f(e)$ implies $a = e$.

$\Longleftarrow$: Suppose $f(a) = f(b)$. Then $a^{-1} * b$ lies in $\ker(f) = \{e\}$. I.e., $a^{-1} * b = e$. Means $a = b$)

**EX** sign: $S_n \longrightarrow \{\pm 1\}$ has $\mathrm{im}(\mathrm{sign}) = \{\pm 1\}$

& $\ker(\mathrm{sign}) = A_n$.

**EX** det: $O_n(\mathbb{R}) \longrightarrow \mathbb{R}^\times$ has $\mathrm{im}(\det) = \{\pm 1\}$

— can restrict to $f: D_N \to \{\pm 1\}$ & $\ker(\det) = SO_n(\mathbb{R})$

which has $\ker(f) = \langle r \rangle =$ rotations.

**EX** exp: $\mathbb{R} \longrightarrow \mathbb{R}^\times$ has $\mathrm{im}(\exp) = (0,\infty) = \mathbb{R}^{>0}$.

& $\ker(\exp) = \{0\}$.

(gives isomorphism:

$$\mathbb{R} \xrightarrow{\sim} \mathbb{R}^{>0}$$

add. mult.

with inverse $\ln(\cdot)$.)

we "polar form"

$\downarrow$

has $\mathrm{im}(\exp) = \mathbb{C}^\times$ (exc.)

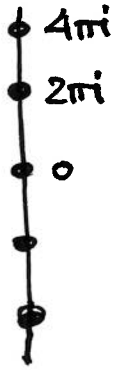
**EX** exp: $\mathbb{C} \longrightarrow \mathbb{C}^\times$ & $\ker(\exp) = 2\pi i\mathbb{Z}$.

$\uparrow$ must have $a=0,\ b \in 2\pi\mathbb{Z}$

— Recall, the complex exponential:

$$\exp(z) = e^a\left(\cos(b) + i\sin(b)\right)$$

$\forall z = a+ib \in \mathbb{C}$.



**EX** $f: \mathbb{R}^\times \longrightarrow \mathbb{R}^\times$

(fix an $N>0$) $f(a) = a^N$ hom.

$$\mathrm{im}(f) = \begin{cases} \mathbb{R}^{>0} & N \text{ even} \\ \mathbb{R}^\times & N \text{ odd} \end{cases}$$

$$\ker(f) = \begin{cases} \{\pm 1\} & N \text{ even} \\ \{1\} & N \text{ odd} \end{cases}$$

(so f is an isomorphism $\mathbb{R}^\times \xrightarrow{\sim} \mathbb{R}^\times$ when N odd) "automorphism"

(keep fixed $N > 0$)

<u>EX</u> $f: \mathbb{C}^\times \longrightarrow \mathbb{C}^\times$

$z \longmapsto z^N$

Here $\mathrm{im}(f) = \mathbb{C}^\times$

& $\ker(f) = U_N$

($\forall w$: can solve $z^N = w$ for $z$)

($N^{th}$ roots of unity)

<u>EX</u> $f: \mathbb{Z}_6 \longrightarrow \mathbb{Z}_6$ "multiplication by [3]".

$$\begin{aligned} [1] &\longmapsto [3] \\ [2] &\longmapsto [0] \\ [3] &\longmapsto [3] \\ [4] &\longmapsto [0] \\ [5] &\longmapsto [3] \end{aligned}$$

$\mathrm{im}(f) = \{[0],[3]\} = \langle [3] \rangle$

& $\ker(f) = \{[0],[2],[4]\}$

$= \langle [2] \rangle$.

(and of course $[0] \longmapsto [0]$)

<u>EX</u> $f: \mathbb{Z}_5^\times \longrightarrow \mathbb{Z}_{\substack{\cancel{12}\\ 6}}$ <u>the</u> homomorphism s.t.

$$f(\underset{\nwarrow \text{ order} = 4.}{[2]}) = [3]$$

(<u>note</u>: $4 \cdot [3] = [0]$)

$$\begin{aligned} [1] &\longmapsto [0] \\ [2] &\longmapsto [3] \\ [3] &\longmapsto [3]. \\ [4] &\longmapsto [0]. \end{aligned}$$

$2^1 \equiv 2$, $2^2 \equiv 4$, $2^3 \equiv 3$.

— For instance, $[3] = [2]^3$ must map to $3 \cdot f([2]) = 3 \cdot [3]$

$= [9]$

$= [3]$.

shows / add.

& $\mathrm{im}(f) = \{[0],[3]\} \leq \mathbb{Z}_6$.

$\ker(f) = \{[1],[4]\} \leq \mathbb{Z}_5^\times$.

mult.