LECTURE 24
(Wednesday Nov. 27, 2019)
Def
(1) The \textbf{image} of $f$ is
\[ \text{im}(f) = \{ y \in H : y = f(x) \text{ for some } x \in G \} \]
\[ = \{ f(x) : x \in G \} \]
\[ = f(G). \]

(2) The \textbf{kernel} of $f$ is
\[ \text{ker}(f) = \{ x \in G : f(x) = e \} = f^{-1}([e]) \]

\textbf{Lemma}:
- $\text{im}(f)$ is a subgroup of $H$.
- $\text{ker}(f)$ is a normal subgroup of $G$.

\textbf{Proof}:
- $\text{im}(f)$ closed under $\cdot$:
\[ f(a) \cdot f(b) = f(x) \text{ some } x \in G, \text{ namely } x = a \cdot b. \]
\[ e_H \in \text{im}(f); \text{ indeed } e_H = f(e_G). \]
\[ \text{im}(f) \text{ closed under inversion by (ii) above}. \]
- $\text{ker}(f)$ closed under $\ast$:
\[ f(a) = e \land f(b) = e \Rightarrow f(a \ast b) = e \ast e = e. \]
\[ e_G \in \text{ker}(f) \checkmark \text{ closed under } a \mapsto a^{-1} \checkmark \]
- (check both as an exc.)
The key point is \( \ker(f) \triangleleft G \) ("normality"): 

Amounts to: \( \forall a \in \ker(f), \forall g \in G \) must check that \( g \ast a \ast g^{-1} \in \ker(f) \).

\[
f(g \ast a \ast g^{-1}) = f(g) \cdot f(a) \cdot f(g^{-1}) = e_H
\]

Remark: \( \text{im}(f) \) not always normal in \( H \).

Ex: \( f: \mathbb{Z}_3 \rightarrow S_3 \)

\[
\begin{align*}
\ker(f) &= \{[0]_3\}^4 \\
\text{im}(f) &= \langle (123) \rangle \\
\end{align*}
\]

\( \not\text{not normal in } S_4: \)

\[
(14) \circ (123) \circ (14)^{-1} = (423) \not\in \text{im}(f).
\]

\[
\uparrow
\]
\[
\text{im}(f).
\]

Observe:

\( \circ \) surjective \iff \( \text{im}(f) = H \)

\( \circ \) injective \iff \( \ker(f) = \{e\} \)

(why? \( \Rightarrow \): \( f(a) = e = f(e) \) implies \( a = e \).

\( \Leftarrow \): Suppose \( f(a) = f(b) \). Then \( a^{-1} \ast b \) lies in \( \ker(f) = \{e\} \). I.e.,

\( a^{-1} \ast b = e \). Means \( a = b \)
**Exercise:** \( \text{sign}: S_n \to \{ \pm 1 \} \) has \( \text{im}(\text{sign}) = \{ \pm 1 \} \) & \( \text{ker}(\text{sign}) = A_n \).

**Exercise:** \( \text{det}: O_n(\mathbb{R}) \to \mathbb{R}^\times \) has \( \text{im}(\text{det}) = \{ \pm 1 \} \) & can restrict to \( f: D_n \to \{ \pm 1 \} \) & \( \text{ker}(\text{det}) = SO_n(\mathbb{R}) \) which has \( \text{ker}(f) = \langle r \rangle = \text{rotations} \).

**Exercise:** \( \exp: \mathbb{R} \to \mathbb{R}^\times \) has \( \text{im}(\exp) = (0, \infty) = \mathbb{R}^> \) & \( \text{ker}(\exp) = \{ 0 \} \).

(\text{gives isomorphism}):

\[ \mathbb{R} \overset{\exp}{\longrightarrow} \mathbb{R}^> \]

\( \sim \) \text{mult.}

\( \text{add.} \)

with inverse \( \ln(x) \).

**Exercise:** \( \exp: \mathbb{C} \to \mathbb{C}^\times \) has \( \text{im}(\exp) = \mathbb{C}^\times \) (exc.) & \( \text{ker}(\exp) = 2\pi i \mathbb{Z} \).

Recall, the complex exponential:

\[ \exp(z) = e^{a \left( \cos(b) + i \sin(b) \right)} \]

\( \forall z = a + ib \in \mathbb{C} \).

**Exercise:** \( f: \mathbb{R}^\times \to \mathbb{R}^\times \)

\( f(1) = 1 \)

\( f(x) = x^N \)

\( \text{im}(f) = \begin{cases} \mathbb{R}^> & N \text{ even} \\ \mathbb{R}^\times & N \text{ odd} \end{cases} \)

\( \text{ker}(f) = \begin{cases} \{ \pm 1 \} & N \text{ even} \\ \{ 1 \} & N \text{ odd} \end{cases} \)

\( \text{(so } f \text{ is an isomorphism} \) \( \mathbb{R}^\times \overset{\sim}{\longrightarrow} \mathbb{R}^\times \) when \( N \text{ odd} \) \)

\( \text{\textquote{T}automorphism\textquote{}} \)
\[ \begin{align*}
\text{Ex. } f : \mathbb{C}^x & \rightarrow \mathbb{C}^x \\
z & \mapsto z^n
\end{align*} \]

Here \( \text{im}(f) = \mathbb{C}^x \) (can solve \( z^n = w \) for \( z \))

\[ \begin{align*}
\text{Ex. } f : \mathbb{Z}_6 & \rightarrow \mathbb{Z}_6 \\
[2] & \mapsto [0] \\
[4] & \mapsto [0] \\
\end{align*} \]

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AND OF COURSE [0] \rightarrow [0]
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EX. f : \mathbb{Z}_5 \rightarrow \mathbb{Z}_{12}, \quad \text{the homomorphism s.t.}
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\[ f([2]) = [3] \]

\[ \leftarrow \text{order} = 4. \]

\[ (\text{note: } 4 \cdot [3] = [0]) \]

\[ 2^1 \equiv 2, \quad 2^2 \equiv 4, \quad 2^3 \equiv 3. \]

\[ \text{For instance, } [3] = [2]^3 \]

须为 \( 3 \cdot f([2]) = 3 \cdot [3] = [9] = [3] \).