LECTURE 5

(Monday OCT. 7, 2019)

Composition Tables: $(G,*)$ finite group.

Choose an ordering of its elements:

$$G = \{x_1, x_2, \dots, x_n\}, \quad n = |G|.$$

the law $*$ is completely described by the table:

$$\begin{array}{c|cccc}
* & x_1 & x_2 & \cdots & x_n \\
\hline
x_1 & & & & \\
x_2 & & & & \boxed{x_2 * x_n} \\
\vdots & & & & \\
x_n & & & &
\end{array}$$

EX G is abelian $\Longleftrightarrow$ table is symmetric.
$(x_i * x_j = x_j * x_i)$

— typically one takes the first element to be $x_1 = e$.

(in which 1^{st} row is simply

$x_1 \quad x_2 \quad \cdots \quad x_n$

similarly for 1^{st} column)

Ex $\mathbb{Z}_3 = \{[0],[1],[2]\}$ with $+$.

$+$	$[0]$	$[1]$	$[2]$
$[0]$	$[0]$	$[1]$	$[2]$
$[1]$	$[1]$	$[2]$	$[0]$
$[2]$	$[2]$	$[0]$	$[1]$

$(\varphi(10) = 4)$

Ex $\mathbb{Z}_{10}^{\times} = \{[1],[3],[7],[9]\}$ with $\cdot$

$\cdot$	$[1]$	$[3]$	$[7]$	$[9]$
$[1]$	$[1]$	$[3]$	$[7]$	$[9]$
$[3]$	$[3]$	$[9]$	$[1]$	$[7]$
$[7]$	$[7]$	$[1]$	$[9]$	$[3]$
$[9]$	$[9]$	$[7]$	$[3]$	$[1]$

Ex $\mathbb{Z}_{12}^{\times} = \{[1], [5], [7], [11]\}$ with $\bullet$ $(\varphi(12) = 4)$

$\bullet$	[1]	[5]	[7]	[11]
[1]	[1]	[5]	[7]	[11]
[5]	[5]	[1]	[11]	[7]
[7]	[7]	[11]	[1]	[5]
[11]	[11]	[7]	[5]	[1]

Note: $\mathbb{Z}_{12}^{\times}$ has the property that every element sat. $x^2 = e$. This is not the case for $\mathbb{Z}_{10}^{\times}$ (only $x = [1]$ and $x = [9]$ sat. $x^2 = e$) — so although $|\mathbb{Z}_{12}^{\times}| = |\mathbb{Z}_{10}^{\times}| = 4$, they're genuinely different groups ("NON-ISOMORPHIC")

Thm ("CANCELLATION LAWS") Let $(G, *)$ be any group. Then $\forall a, b, c \in G$:

(i) $a * b = a * c \implies b = c$

(ii) $b * a = c * a \implies b = c$.

Why? We'll only do (i).

a has an inverse a^{-1}. Scale both sides of the eqn. $a*b=a*c$ by a^{-1} on the left:

$$\begin{array}{ccc}
a^{-1}*(a*b) & = & a^{-1}*(a*c) \\
\| & & \| \quad \text{"associative law"} \\
(a^{-1}*a)*b & & (a^{-1}*a)*c \\
\| & & \| \quad \text{"inverse"} \\
e*b & & e*c \\
\| & & \| \quad \text{"neutral"} \\
b & & c.
\end{array}$$

Reads $\boxed{b=c}$. Done ✓

(*) WARNING: "a" must be on the same side!
(when G non-abelian)

EX: $G = GL_2(\mathbb{R})$ w. matrix multiplication.

Take $a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $c = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

Then: $ab = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = ca$, but $b \neq c$.

[So: $ab = ca \Rightarrow b = c$ is FALSE in general]

∘ Interpretation: Each $a \in G$ defines a function

"left multiplication by a".

$$f_a : G \longrightarrow G$$
$$x \longmapsto a * x$$

The cancellation law says f_a is injective. I.e.,

$$f_a(x) = f_a(y) \implies x = y.$$

It's obviously also surjective: Given any $g \in G$, we may express it as $g = f_a(x)$ where $x = a^{-1} * g$.

Corollary: f_a is bijective $(\forall a \in G)$.

— in other words: Multiplication by "a" permutes the elements of G.

~ each row & column of the composition table contains each element of G precisely once.

EXC: $(a, b \in G)$

$$\underbrace{f_a \circ f_b}_{\text{composition of functions}} = \underbrace{f_{a*b}}_{\text{group law}}$$

— cf. CAYLEY'S THEOREM. (later..)