Math 103A, Modern Algebra I, MT1

Monday, October 21st, 2019, 10-10:50am, APM B402A

• Your Name:
• ID Number:
• Section:

B01 (5:00 PM)  B02 (6:00 PM)

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<th>Problem #</th>
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Total (out of 40):
Problem 1. Recall that \( \mathbb{Z}_{13} \) denotes the additive group of all residue classes modulo 13.

(a) Find its cardinality \( |\mathbb{Z}_{13}| \).

(b) Compute the following sums in \( \mathbb{Z}_{13} \).

\[
[7] + [11], \quad [11] + [12], \quad [-3] + [17].
\]

(Express your answers in the form \([x]\) with \( x \) in the range \( 0 \leq x < 13 \).)

(c) Give the inverse class of \([6]\) in \( \mathbb{Z}_{13} \). More precisely find the \( x \in \mathbb{Z} \) in the range \( 0 \leq x < 13 \) satisfying the congruence \( 6 + x \equiv 0 \pmod{13} \).
Problem 2. Recall that $\mathbb{Z}_{13}^\times$ denotes the multiplicative group of invertible residue classes modulo 13.

(a) Find its cardinality $|\mathbb{Z}_{13}^\times|$.

(b) Compute the following products in $\mathbb{Z}_{13}^\times$.

$$[3] \cdot [5], \quad [5] \cdot [7], \quad [7] \cdot [-11].$$

(Express your answers in the form $[x]$ with $x$ in the range $0 \leq x < 13$.)

(c) Explain why $[11]$ belongs to $\mathbb{Z}_{13}^\times$ and find its inverse class: Find $x \in \mathbb{Z}$ in the range $0 \leq x < 13$ satisfying the congruence $11x \equiv 1 \pmod{13}$. 

**Problem 3.** Let $F$ denote the set of all numbers of the form $a + b\sqrt{3}$ with rational coefficients $a, b \in \mathbb{Q}$. Let $F^\times = F - \{0\}$ be the nonzero numbers.

(a) Explain why addition and multiplication $\bullet$ define composition laws on $F$.

(b) Check that $(F, +)$ and $(F^\times, \bullet)$ are groups. Give all details.

(c) Compute the multiplicative inverse of $1 + 2\sqrt{3}$. Find $a, b \in \mathbb{Q}$ such that

$$(1 + 2\sqrt{3})^{-1} = a + b\sqrt{3}.$$
Problem 4. Let \((G, *)\) be a group. Two elements \(a, b \in G\) are \emph{conjugate} when one can write \(b = g * a * g^{-1}\) for some \(g \in G\). If so we write \(a \sim b\).

(a) Find all \(a \in G\) satisfying \(a \sim e\) (where \(e\) is the neutral element).

(b) Prove that \(\sim\) gives an equivalence relation on \(G\). Check these properties:

(i) \(a \sim a\)

(ii) \(a \sim b \iff b \sim a\)

(iii) \(a \sim b\) and \(b \sim c \implies a \sim c\)

for any three elements \(a, b, c \in G\).

(c) Are the two matrices \(\begin{pmatrix} 1 & 2 \\ 3 & \pi \end{pmatrix}\) and \(\begin{pmatrix} \pi & 1 \\ 2 & 3 \end{pmatrix}\) conjugate in \(\text{GL}_2(\mathbb{R})\)?