Math 103A, Modern Algebra I, MT1

Monday, October 21st, 2019, 10-10:50am, APM B402A

- *Your Name*:
- *ID Number*:
- *Section*:

B01 (5:00 PM) B02 (6:00 PM)

Problem #	Points (out of 10)
1	
2	
3	
4	
Total (out of 40):	

Problem 1. Recall that $\mathbb{Z}_{13}$ denotes the additive group of all residue classes modulo 13.

(a) Find its cardinality $|\mathbb{Z}_{13}|$.

(b) Compute the following sums in $\mathbb{Z}_{13}$.

$$[7] + [11], \qquad [11] + [12], \qquad [-3] + [17].$$

(Express your answers in the form $[x]$ with x in the range $0 \leq x < 13$.)

(c) Give the inverse class of $[6]$ in $\mathbb{Z}_{13}$. More precisely find the $x \in \mathbb{Z}$ in the range $0 \leq x < 13$ satisfying the congruence $6 + x \equiv 0 \pmod{13}$.

Problem 2. Recall that $\mathbb{Z}_{13}^{\times}$ denotes the multiplicative group of invertible residue classes modulo 13.

(a) Find its cardinality $|\mathbb{Z}_{13}^{\times}|$.

(b) Compute the following products in $\mathbb{Z}_{13}^{\times}$.

$$[3] \bullet [5], \qquad [5] \bullet [7], \qquad [7] \bullet [-11].$$

(Express your answers in the form $[x]$ with x in the range $0 \leq x < 13$.)

(c) Explain why [11] belongs to $\mathbb{Z}_{13}^{\times}$ and find its inverse class: Find $x \in \mathbb{Z}$ in the range $0 \leq x < 13$ satisfying the congruence $11x \equiv 1 \pmod{13}$.

Problem 3. Let F denote the set of all numbers of the form $a + b\sqrt{3}$ with rational coefficients $a, b \in \mathbb{Q}$. Let $F^\times = F - \{0\}$ be the nonzero numbers.

(a) Explain why addition and multiplication $\bullet$ define composition laws on F.

(b) Check that $(F, +)$ and $(F^\times, \bullet)$ are groups. Give all details.

(c) Compute the multiplicative inverse of $1 + 2\sqrt{3}$. Find $a, b \in \mathbb{Q}$ such that

$$(1 + 2\sqrt{3})^{-1} = a + b\sqrt{3}.$$

Problem 4. Let $(G, *)$ be a group. Two elements $a, b \in G$ are *conjugate* when one can write $b = g * a * g^{-1}$ for some $g \in G$. If so we write $a \sim b$.

(a) Find all $a \in G$ satisfying $a \sim e$ (where e is the neutral element).

(b) Prove that $\sim$ gives an equivalence relation on G. Check these properties:

  (i) $a \sim a$

  (ii) $a \sim b \Longleftrightarrow b \sim a$

  (iii) $a \sim b$ and $b \sim c \Longrightarrow a \sim c$

  for any three elements $a, b, c \in G$.

(c) Are the two matrices $\left(\begin{smallmatrix} 1 & 2 \\ 3 & \pi \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \pi & 1 \\ 2 & 3 \end{smallmatrix}\right)$ conjugate in $\mathrm{GL}_2(\mathbb{R})$?