Math 103B, Winter 2018

Modern Algebra II, HW 1

Due Friday January 19th by 5PM in your TA's box

From Lauritzen's book:

- Exercises 3.6 (starting page 138): *3*[1], *1, 2, 15, 23*

Problem A. In this exercise let $\iota = \sqrt{-1}$ and consider the four matrices

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad i = \begin{pmatrix} \iota & 0 \\ 0 & -\iota \end{pmatrix} \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad k = \begin{pmatrix} 0 & \iota \\ \iota & 0 \end{pmatrix}.$$

Let $\mathbb{H} \subset M_2(\mathbb{C})$ denote the set of all matrices $q = a1 + bi + cj + dk$ where the coefficients run over all real numbers ($a, b, c, d \in \mathbb{R}$).

(a) Check the relations $i^2 = j^2 = k^2 = ijk = -1$ and conclude that $\mathbb{H}$ is a subring of $M_2(\mathbb{C})$; the ring of quaternions. Is it commutative?

(b) Define the conjugate of the quaternion q above to be $\bar{q} = a1 - bi - cj - dk$. Verify the relations

$$q\bar{q} = \bar{q}q = \|q\|^2 \cdot 1 \quad \text{where} \quad \|q\| = \sqrt{a^2 + b^2 + c^2 + d^2}.$$

(c) Explain why every $q \neq 0$ is a unit and give a formula for its inverse q^{-1}.

(d) Observe that $\|q\| = \sqrt{\det(q)}$ and deduce that the norm is multiplicative:

$$\|pq\| = \|p\| \cdot \|q\| \quad \forall p, q \in \mathbb{H}.$$

Problem B. (Continuation of Problem A.) Let $\mathcal{O}$ be the set of all quaternions $q = a1 + bi + cj + dk$ with integer coefficients ($a, b, c, d \in \mathbb{Z}$).

(a) Show that $\mathcal{O}$ is a subring of $\mathbb{H}$.

(b) Prove that q is a unit of $\mathcal{O}$ if and only if $\|q\| = 1$. (`Hint`: Use A(d).)

(c) Conclude that $\mathcal{O}^\times = \{\pm 1, \pm i, \pm j, \pm k\}$. (cf. Exc. 16 on p. 105.)

[1] Do number 3 first – you will need the result to do exercise 1 next.

Problem C. Let G be a finite group, and let $\mathbb{C}[G]$ denote the set of all functions $f : G \to \mathbb{C}$. Addition on $\mathbb{C}[G]$ is defined by $(f + g)(x) = f(x) + g(x)$, whereas multiplication is given by so-called convolution:

$$(f \star g)(x) := \sum_{t \in G} f(t)g(t^{-1}x) = \sum_{t \in G} f(xt^{-1})g(t).$$

(a) Check the second equality in the line above. (`Hint`: Substitute $\tau = t^{-1}x$.)

(b) For each $a \in G$ let δ_a denote the function with $\delta_a(a) = 1$ and $\delta_a(x) = 0$ for all $x \neq a$. Prove the relation

$$\delta_a \star \delta_b = \delta_{ab} \quad \forall a, b \in G$$

(c) Infer from (b) that $\mathbb{C}[G]$ is a ring. What is its multiplicative identity $1_{\mathbb{C}[G]}$? (`Hint`: Any f can be expanded as a linear combination $f = \sum_{a \in G} f(a)\delta_a$.)