From Lauritzen’s book:

- Exercises 3.6 (starting page 138): 4, 9, 11, 12, 14, 26

**Problem A.** Let \( R \) and \( S \) be any two rings. Consider the direct product

\[ R \times S = \{ (r, s) : r \in R, \ s \in S \} \]

equipped with componentwise addition and multiplication. That is,

\[ \begin{align*}
(r, s) + (r', s') &= (r + r', s + s'), \\
(r, s) \cdot (r', s') &= (rr', ss').
\end{align*} \]

(a) Verify that \( R \times S \) is a ring, and that it is commutative precisely when both \( R \) and \( S \) are commutative. Find the neutral elements \( 0_{R \times S} \) and \( 1_{R \times S} \).

(b) Check that \((r, s)\) is a unit of \( R \times S \) if and only if \( r \in R \times \) and \( s \in S \times \).

(c) Explain why \( R \times S \) has zero-divisors if \( R \) and \( S \) are nonzero rings.

(d) Let \( \pi : R \times S \to R \) be the projection map \( \pi((r, s)) = r \). Show that \( \pi \) is a surjective ring homomorphism and find its kernel.

(e) Suppose \( R \) and \( S \) both have positive characteristic. Prove that

\[ \text{char}(R \times S) = \text{LCM}(\text{char}(R), \text{char}(S)). \]

**Problem B.** Let \( R \) be a domain, and let \( S \subset R \setminus \{0\} \) be a multiplicative subset. (That is \( S \) is closed under multiplication and \( 1 \in S \).) Introduce the following subset of the fraction field.

\[ S^{-1}R = \{ \frac{r}{s} : r \in R, \ s \in S \}. \]

(a) Check that \( S^{-1}R \) is a subring of the fraction field \( \text{Frac}(R) \).
(b) Let \( i : R \to S^{-1}R \) be the homomorphism \( i(r) = \frac{r}{1} \). Show that all elements of the form \( i(s) = \frac{s}{1} \) with \( s \in S \) are units in \( S^{-1}R \).

(c) Suppose \( \varphi : R \to R' \) is a homomorphism for which \( \varphi(S) \subset (R')^\times \). Prove that there is a unique homomorphism \( \bar{\varphi} : S^{-1}R \to R' \) such that \( \varphi = \bar{\varphi} \circ i \).