Math 103B, Winter 2018

Modern Algebra II, HW 2

Due Friday January 26th by 5PM in your TA's box

From Lauritzen's book:

- Exercises 3.6 (starting page 138): *4, 9, 11, 12, 14, 26*

Problem A. Let R and S be any two rings. Consider the direct product

$$R \times S = \{(r,s) : r \in R, \quad s \in S\}$$

equipped with componentwise addition and multiplication. That is,

- $(r,s) + (r',s') = (r+r', s+s')$,
- $(r,s) \cdot (r',s') = (rr', ss')$.

(a) Verify that $R \times S$ is a ring, and that it is commutative precisely when both R and S are commutative. Find the neutral elements $0_{R\times S}$ and $1_{R\times S}$.

(b) Check that (r,s) is a unit of $R \times S$ if and only if $r \in R^\times$ and $s \in S^\times$.

(c) Explain why $R \times S$ has zero-divisors if R and S are nonzero rings.

(d) Let $\pi : R \times S \to R$ be the projection map $\pi\big((r,s)\big) = r$. Show that π is a surjective ring homomorphism and find its kernel.

(e) Suppose R and S both have positive characteristic. Prove that

$$\mathrm{char}(R \times S) = \mathrm{LCM}\big(\mathrm{char}(R), \mathrm{char}(S)\big).$$

Problem B. Let R be a domain, and let $S \subset R\backslash\{0\}$ be a multiplicative subset. (That is S is closed under multiplication and $1 \in S$.) Introduce the following subset of the fraction field.

$$S^{-1}R = \{\frac{r}{s} : r \in R, \quad s \in S\}.$$

(a) Check that $S^{-1}R$ is a subring of the fraction field $\mathrm{Frac}(R)$.

(b) Let $i : R \to S^{-1}R$ be the homomorphism $i(r) = \frac{r}{1}$. Show that all elements of the form $i(s) = \frac{s}{1}$ with $s \in S$ are units in $S^{-1}R$.

(c) Suppose $\varphi : R \to R'$ is a homomorphism for which $\varphi(S) \subset (R')^{\times}$. Prove that there is a unique homomorphism $\bar{\varphi} : S^{-1}R \to R'$ such that $\varphi = \bar{\varphi} \circ i$.