Math 103B, Winter 2018

Modern Algebra II, HW 3

Due Friday February 2nd by 5PM in your TA's box

From Lauritzen's book:

- Exercises 3.6 (starting page 138): *5, 8, 10, 16, 17*

Problem A. We introduce the following subset of complex numbers

$$\mathbb{Z}[\sqrt{-5}] = \{a + b\sqrt{-5} : a, b \in \mathbb{Z}\}.$$

(a) Check that $\mathbb{Z}[\sqrt{-5}]$ is a subring of $\mathbb{C}$ and find its units.

(b) Consider the principal ideal $(1 + \sqrt{-5})$. Show that the unique ring homomorphism

$$\varphi : \mathbb{Z} \longrightarrow \mathbb{Z}[\sqrt{-5}]/(1 + \sqrt{-5})$$

given by $\varphi(a) = a + (1 + \sqrt{-5})$ is surjective.

(c) Prove that $\ker(\varphi) = 6\mathbb{Z}$ and conclude that $\mathbb{Z}[\sqrt{-5}]/(1 + \sqrt{-5}) \simeq \mathbb{Z}/6\mathbb{Z}$.

Problem B. Let $\mathcal{C}(\mathbb{R})$ be the ring of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ with addition and multiplication defined pointwise. I.e.,

$$(f + g)(x) = f(x) + g(x) \qquad (fg)(x) = f(x)g(x).$$

(a) For every subset $X \subset \mathbb{R}$ consider the subset of functions vanishing on X,

$$I_X = \{f \in \mathcal{C}(\mathbb{R}) : f(x) = 0 \ \ \forall x \in X\}.$$

Show that I_X is an ideal of $\mathcal{C}(\mathbb{R})$.

(b) Suppose we have an inclusion $Y \subset X$. Explain why $I_X \subset I_Y$, and use this observation to construct an increasing chain of ideals

$$I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots \subsetneq I_i \subsetneq \cdots.$$

Conclude that $\mathcal{C}(\mathbb{R})$ is not a noetherian ring (cf. exc. 3.6.10(i)).

(c) Take X to be a singleton (i.e., a set containing exactly one real number). Prove that the quotient ring $\mathcal{C}(\mathbb{R})/I_X$ is isomorphic to $\mathbb{R}$. Conclude that I_X is a maximal ideal.

Problem C. Let R and S be commutative rings, and consider the product ring $R \times S$ (cf. Problem A on HW2). Let $\pi_R : R \times S \to R$ be the projection map given by $\pi_R\big((r,s)\big) = r$ and define $\pi_S : R \times S \to S$ analogously.

(a) Let $I \subset R \times S$ be an ideal. Show that its two images $\pi_R(I)$ and $\pi_S(I)$ are ideals in R and S respectively. (For example, $\pi_R(I)$ consists of all $r \in R$ such that $(r, \star) \in I$ for some $\star \in S$.)

(b) Prove that $I = \pi_R(I) \times \pi_S(I)$. (The key point is the inclusion $\supset$ which uses the ideal property of I. `Hint`: $(r,s) = (1,0)(r,\star) + (0,1)(*,s)$.)

(c) Deduce that there is an isomorphism of rings

$$(R \times S)/I \xrightarrow{\sim} R/\pi_R(I) \times S/\pi_S(I).$$