Math 103B, Winter 2018

Modern Algebra II, HW 4

Due Friday February 9th by 5PM in your TA's box

From Lauritzen's book:

- Exercises 3.6 (starting page 138): *19, 20, 22, 25, 39*

Problem A. Let R be a commutative ring, and $I \subset R$ an ideal. Introduce

$$\sqrt{I} := \{r \in R : r^n \in I \text{ for some } n \geq 1\}.$$

(a) Prove that $\sqrt{I}$ is an ideal of R. (`Hint`: To prove closure under addition use the binomial formula, observing that $r^n \in I$ for all large enough n.)

(b) Check that $I \subset \sqrt{I}$. If R is noetherian prove the inclusion $\sqrt{I}^m \subset I$ for some positive integer m. (Here $\sqrt{I}^m = \sqrt{I} \cdots \sqrt{I}$ with m factors.)

(c) Show that I is a radical ideal (meaning $I = \sqrt{I}$) if and only if R/I is reduced (i.e., has no nonzero nilpotent[1] elements); verify that prime ideals are radical.

(d) Take $R = \mathbb{Z}$ and let $N \geq 1$. Prove that $\sqrt{N\mathbb{Z}} = \tilde{N}\mathbb{Z}$ where $\tilde{N}$ is the largest squarefree divisor of N. (More concretely, $\tilde{N}$ is obtained from the prime factorization of N by replacing all exponents by one.)

Problem B. A *local* ring is a commutative ring with a unique maximal ideal.

(a) Explain why fields and $\mathbb{Z}/p^n\mathbb{Z}$ for primes p are examples of local rings.

(b) Suppose R is a commutative ring which admits an ideal $\mathfrak{m} \neq R$ such that $R^\times = R \backslash \mathfrak{m}$ (the set-theoretic complement of $\mathfrak{m}$). Prove that R is a local ring and $\mathfrak{m}$ is its unique maximal ideal.

(c) Conversely, assume R is a noetherian local ring with maximal ideal $\mathfrak{m}$. Show that $R^\times = R \backslash \mathfrak{m}$.

Problem C. Let R, S be commutative rings and $\varphi : R \to S$ a homomorphism.

[1] An element $r \in R$ is *nilpotent* if $r^n = 0$ for some $n \geq 1$.

(a) For an ideal $I \subset S$ consider its inverse image in R defined as the subset

$$\varphi^{-1}(I) = \{r \in R : \varphi(r) \in I\}.$$

Check that $\varphi^{-1}(I)$ is an ideal of R. Furthermore, show that there is an injective homomorphism of rings

$$R/\varphi^{-1}(I) \longrightarrow S/I$$

and deduce that $R/\varphi^{-1}(I)$ is isomorphic to a subring of S/I.

(b) Give an example showing the image of an ideal of R is not necessarily an ideal of S. What if we assume φ is surjective; is the image of an ideal an ideal?

(c) Suppose I is a prime ideal. Explain why $\varphi^{-1}(I)$ must be a prime ideal.

(d) Give an example showing that I maximal does not imply $\varphi^{-1}(I)$ is maximal. (`Hint`: $\mathbb{Z}$ in $\mathbb{Q}$.)