Math 103B, Winter 2018

Modern Algebra II, HW 5

Due Friday February 16th by 5PM in your TA's box

From Lauritzen's book:

- Exercises 3.6 (starting page 138): *27, 28, 32, 35, 36*

Problem A. Let $\mathbb{Z}[\sqrt{-10}] = \{a + b\sqrt{-10} : a, b \in \mathbb{Z}\}$.

(a) Verify that $\mathbb{Z}[\sqrt{-10}]$ is a subring of $\mathbb{C}$ and find its units.

(b) Explain in detail why the fraction field of $\mathbb{Z}[\sqrt{-10}]$ is isomorphic to

$$\mathbb{Q}(\sqrt{-10}) = \{a + b\sqrt{-10} : a, b \in \mathbb{Q}\}.$$

(c) Show that $2 + \sqrt{-10}$ is an irreducible element of $\mathbb{Z}[\sqrt{-10}]$.

(d) Prove that $\mathbb{Z}/14\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}[\sqrt{-10}]/(2 + \sqrt{-10})$.

(e) Deduce from (d) that $2 + \sqrt{-10}$ is not a prime element of $\mathbb{Z}[\sqrt{-10}]$.

Problem B. Introduce the complex number $\theta = \frac{1+\sqrt{-7}}{2} = \frac{1}{2}(1 + i\sqrt{7})$.

(a) Check that $\theta^2 - \theta + 2 = 0$.

(b) Show that $\mathbb{Z}[\theta] = \{a + b\theta : a, b \in \mathbb{Z}\}$ is a subring of $\mathbb{C}$. Find $\mathbb{Z}[\theta]^\times$.

(c) Express the norm $N(a + b\theta) = |a + b\theta|^2$ as a quadratic form in a, b.

(d) Prove that $\mathbb{Z}[\theta]/(\theta)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$; conclude θ is a prime element.

Problem C. A commutative ring R is said to be *Artinian* if every decreasing chain of ideals

$$I_1 \supset I_2 \supset I_3 \supset \cdots$$

stabilizes. That is, $I_i = I_{i+1}$ for all sufficiently large indices $i \geq 1$.

(a) Show that every finite ring (such as $\mathbb{Z}/N\mathbb{Z}$) is Artinian, but $\mathbb{Z}$ is not.

(b) Suppose $R \neq \{0\}$ is an Artinian ring. Prove the existence of a nonzero ideal I which is minimal among all the nonzero ideals of R.

(c) Describe[1] the minimal ideals of $\mathbb{Z}/N\mathbb{Z}$. Write them down for $N = 60$.

[1] All ideals have the form $M\mathbb{Z}/N\mathbb{Z}$ for some $M|N$. For which $M < N$ is the ideal minimal?