Math 104A, Fall 2018

Number Theory, HW 3

Due Wednedsday October 24th by 5PM in Shubham Sinha's box.

From Weissman's book `An illustrated theory of numbers`:

- Exercises (Section 2, pages 72–73):

  1, 2, 3, 4, 5, 6, 9, 18[1]

Problem A. As in exercise 18 let e_p denote the exponent of p in the prime factorization of $n! = 1 \cdot 2 \cdot 3 \cdots n$.

(a) Show that among $\{1, 2, 3, \ldots, n\}$ there are exactly $\left[\frac{n}{p^r}\right]$ multiples of p^r. Here the brackets $[\bullet]$ means taking the floor – and r is any positive integer.

(b) Infer that among $\{1, 2, 3, \ldots, n\}$ there are exactly $\left[\frac{n}{p^r}\right] - \left[\frac{n}{p^{r+1}}\right]$ numbers with exactly r factors of p in the prime factorization.

(c) Conclude that de Polignac's formula holds – that is

$$e_p = \sum_{r=1}^{\infty} \left[\frac{n}{p^r}\right].$$

(Why is this actually a finite sum?)

(d) Expanding n in base p as $n = a_0 + a_1 p + a_2 p^2 + \cdots$ with non-negative coefficients $a_i < p$ deduce from (c) that

$$e_p = \frac{n-s}{p-1} \qquad s := a_0 + a_1 + a_2 + \cdots .$$

Problem B. Let p be a prime and r any integer in the range $0 < r < p$. Prove that p divides the binomial coefficient $\binom{p}{r}$. (`Hint`: Apply Euclid's lemma to the identity

$$r!\binom{p}{r} = p(p-1)\cdots(p-r+1).)$$

Give an example showing this fails without the assumption that p is prime.

[1] Do Problem A below first.