Math 104B, Winter 2019

Number Theory II, HW 2

Due Thursday January 24th by 10AM in Shubham Sinha's box.

From Weissman's book `An illustrated theory of numbers`:

- Exercises (Section 4, pages 122–123):

  2, 3, 5, 9, 10, 12, 13

Problem A. Suppose $x, y, z \in \mathbb{Z}$ satisfy the Pythagorean equation $x^2+y^2=z^2$, and that $\text{GCD}(x, y, z) = 1$.

(a) Check that the Pythagorean equation is equivalent to the following identity in $\mathbb{Z}[i]$:

$$(x+iy)(x-iy) = z^2.$$

(b) Explain why the two factors $x \pm iy$ are coprime. (`Hint`: Suppose $\pi \in \mathbb{Z}[i]$ is an irreducible element dividing both. Then $\pi | 2x$ and $\pi | 2iy$. Using that $\text{GCD}(x, y) = 1$ conclude that $\pi \sim 1+i$. Compute the ratio $\frac{x+iy}{1+i}$ and infer that x and y must be odd. Get a contradiction by reducing z^2 modulo 4.)

(c) Using the unique factorization property of $\mathbb{Z}[i]$, deduce from (b) that $x+iy$ is associated to a square. That is, for suitable $r, s \in \mathbb{Z}$ we have

$$x + iy \sim (r + is)^2.$$

(d) Assuming y is **even**, show that $x + iy$ <u>is</u> a square. That is, one can pick $r, s \in \mathbb{Z}$ such that $x + iy = (r + is)^2$.

(e) Conclude that every primitive Pythagorean triple (x, y, z) with $z > 0$ and y even has the form

$$x = r^2 - s^2 \qquad y = 2rs \qquad z = r^2 + s^2$$

for suitable $r, s \in \mathbb{Z}$.