MATH 104B, WINTER 2019

NUMBER THEORY II, HW 3

Due Thursday January 31st by 10AM in Shubham Sinha's box.

From Weissman's book `An illustrated theory of numbers`:

- Exercise (Section 4, page 123):

  16

From Niven, Zuckerman, Montgomery (5th Ed.):

- Problems (Section 9.5, pages 427–428):

  1, 2, 4, 6

From Niven, Zuckerman, Montgomery (5th Ed.):

- Problem (Section 9.6, page 429):

  1

Problem A. Our goal here is to prove Theorem 4.27 in Weissman's book.

(a) Let $p \neq 3$ be a prime number which can be expressed as $p = a^2 - ab + b^2$ with $a, b \in \mathbb{Z}$. Show that $p \equiv 1 \pmod 3$. (`Hint`: Complete the square, multiply by 4, and reduce modulo 3.)

(b) Conversely, let p be a prime number satisfying $p \equiv 1 \pmod 3$.

(i) Use the quadratic reciprocity law to show that there exists an $x \in \mathbb{Z}$ such that

$$x^2 \equiv -3 \pmod p.$$

(ii) Deduce from (i) that p is reducible in the Eisenstein integers $\mathbb{Z}[\omega]$. (`Hint`: Otherwise p has the prime property and therefore divides $x \pm \sqrt{-3}$.)

(iii) Conclude from (ii) that p has the form $a^2 - ab + b^2$ for suitable $a, b \in \mathbb{Z}$.

Problem B. Let $x, y, z \in \mathbb{Z}$ satisfy the equation $x^3+y^3+z^3=0$. This problem shows[1] that $xyz \equiv 0 \pmod 3$.

(a) Verify that $3 \sim \pi^2$ where $\pi = 1-\omega$.

(b) Check that any element of $\mathbb{Z}[\omega]$ is congruent to either 0 or ± 1 modulo π.

(c) Suppose $\theta \in \mathbb{Z}[\omega]$ is <u>not</u> divisible by π. Show that

$$\theta^3 \equiv \pm 1 \pmod{\pi^4}.$$

(`Hint`: Write $\theta^3 - 1 = (\theta-1)(\theta-\omega)(\theta-\bar{\omega})$ and assume $\theta \equiv 1 \pmod \pi$.)

(d) Suppose for the sake of contradiction that $3 \nmid xyz$. Deduce from (c) that

$$0 = x^3+y^3+z^3 \equiv \pm 1 \pm 1 \pm 1 \pmod{\pi^4}$$

for one of the eight sign combinations. Why is this impossible?

[1] In fact $xyz = 0$ but this requires more effort, cf. Section 9.10 in [NZM].