Problem A. Our goal here is to prove Theorem 4.27 in Weissman’s book.

(a) Let $p \neq 3$ be a prime number which can be expressed as $p = a^2 - ab + b^2$ with $a, b \in \mathbb{Z}$. Show that $p \equiv 1 \pmod{3}$. (Hint: Complete the square, multiply by 4, and reduce modulo 3.)

(b) Conversely, let $p$ be a prime number satisfying $p \equiv 1 \pmod{3}$.

(i) Use the quadratic reciprocity law to show that there exists an $x \in \mathbb{Z}$ such that

$$\quad x^2 \equiv -3 \pmod{p}.$$ 

(ii) Deduce from (i) that $p$ is reducible in the Eisenstein integers $\mathbb{Z}[\omega]$.

(Hint: Otherwise $p$ has the prime property and therefore divides $x \pm \sqrt{-3}$.)

(iii) Conclude from (ii) that $p$ has the form $a^2 - ab + b^2$ for suitable $a, b \in \mathbb{Z}$. 
Problem B. Let $x, y, z \in \mathbb{Z}$ satisfy the equation $x^3 + y^3 + z^3 = 0$. This problem shows\(^1\) that $xyz \equiv 0 \pmod{3}$.

(a) Verify that $3 \sim \pi^2$ where $\pi = 1 - \omega$.

(b) Check that any element of $\mathbb{Z}[\omega]$ is congruent to either 0 or $\pm 1$ modulo $\pi$.

(c) Suppose $\theta \in \mathbb{Z}[\omega]$ is not divisible by $\pi$. Show that

$$\theta^3 \equiv \pm 1 \pmod{\pi^4}.$$  

(Hint: Write $\theta^3 - 1 = (\theta - 1)(\theta - \omega)(\theta - \bar{\omega})$ and assume $\theta \equiv 1 \pmod{\pi}$.)

(d) Suppose for the sake of contradiction that $3 \nmid xyz$. Deduce from (c) that

$$0 = x^3 + y^3 + z^3 \equiv \pm 1 \pm 1 \pm 1 \pmod{\pi^4}$$

for one of the eight sign combinations. Why is this impossible?

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\(^1\)In fact $xyz = 0$ but this requires more effort, cf. Section 9.10 in [NZM].