Math 104B, Winter 2019

Number Theory II, HW 6

Due Thursday February 21st by 10AM in Shubham Sinha's box.

From Niven, Zuckerman, Montgomery (5th Ed.):

- Problems (Section 6.2, page 307):

  2, 4

- Problems (Section 6.3, pages 311–312):

  1, 2, 3, 7, 8

Problem A. Our goal is to show $\xi = \sum_{n=1}^{\infty} 3^{-n!}$ is transcendental.

(a) Why is the infinite series defining ξ convergent?

(b) Let $s_N = \sum_{n=1}^{N} 3^{-n!}$ be the Nth partial sum. Check that $s_N = \frac{a}{3^{N!}}$ where $3 \nmid a$.

(c) Observe that for all N sufficiently large we have an inequality

$$|\xi - \frac{a}{3^{N!}}| = \sum_{n=N+1}^{\infty} 3^{-n!} < \frac{1}{3^{(N+1)!}} \cdot \left(1 + \frac{1}{3^{(N+1)}} + \frac{1}{3^{(N+1)(N+2)}} + \cdots\right).$$

Moreover, show that the parenthetical sum is $< \frac{3}{2}$ by comparing it to a geometric sum.

(d) Conclude from (c) that for all N large enough we have

$$|\xi - \frac{a}{3^{N!}}| < \frac{1/2}{(3^{N!})^N}.$$

(e) If ξ is algebraic of degree $m > 1$, Liouville's approximation theorem gives the existence of a constant $c(\xi) > 0$ for which

$$|\xi - \frac{r}{s}| \geq \frac{c(\xi)}{s^m}$$

for **all** $\frac{r}{s} \in \mathbb{Q}$ with $s > 0$. Why does this contradict the inequality in (d)?