Problem A. Let $f \in \mathcal{A}$ be an arbitrary arithmetic function with Möbius transformation $\hat{f}$. Recall that this is defined by the sum $\hat{f}(n) = \sum_{d|n} f(d)$.

(a) Prove the following inequality for all $x > 1$.

$$\left| \frac{1}{x} \sum_{n \leq x} n \hat{f}(n) - \frac{1}{2} \sum_{d \leq x} f(d) \left( \frac{x}{d} + 1 \right) \right| < \sum_{d \leq x} |f(d)|.$$  

(Hint: Mimic how we estimated the sum $\sum_{n \leq x} \hat{f}(n)$.)

(b) By taking the function $f(n) = \frac{\mu(n)}{n}$ in part (a) deduce that the average order of Euler’s $\phi$-function satisfies the asymptotics

$$\frac{1}{x} \sum_{n \leq x} \phi(n) = \frac{3}{\pi^2} x + O(\ln x).$$

(You may use the formula $\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{6}{\pi^2}$ which we still have not shown.)

\footnote{Hint: Try the function $f(n) = \frac{1}{n}$ in (8.48) on page 389.}