Problem A. Let \(a_1, a_2, \ldots, a_n, \ldots\) be a sequence of complex numbers, and let
\[
A(x) = \sum_{n \leq x} a_n
\]
be the summatory function associated with the sequence.

(a) For any function \(f : (0, \infty) \rightarrow \mathbb{C}\), verify the relation below for all \(x > 1\).
\[
\sum_{n \leq x} a_n f(n) = A(x)f([x]) + \sum_{n \leq x-1} A(n)(f(n) - f(n+1))
\]
(Hint: Insert \(a_n = A(n) - A(n-1)\) in the sum on the left-hand side.)

(b) Assuming \(f\) is continuously differentiable on \((0, \infty)\), use the fundamental theorem of calculus to rewrite the relation in part (a) as
\[
\sum_{n \leq x} a_n f(n) = A(x)f(x) - \int_1^x A(t)f'(t)dt.
\]
This identity is known as the "partial summation formula".

(c) Suppose \(A(x) = x + O(x^{1-\epsilon})\) for some constant \(\epsilon > 0\). By taking \(f(t) = t^{-s}\) in part (b) for some fixed \(s \in \mathbb{C}\) with real part \(\text{Re}(s) \leq 1 - \epsilon\), deduce that
\[
\sum_{n \leq x} \frac{a_n}{n^s} = \frac{1}{1-s}x^{1-s} + O(x^{1-s-\epsilon})
\]
as \(x \rightarrow \infty\).