Math 109, Mathematical Reasoning, Final Exam

Monday, March 16th, 2020, 11:30am-2:30pm, TBA

- *Your Name*:
- *ID Number*:
- *Section*:

C01 (4:00 PM)   C02 (5:00 PM)

Problem #	Points (out of 10)
1	
2	
3	
4	
5	
6	
7	
8	
9	
Total (out of 90):	

Problem 1. Which of the four sets below are countable? Justify your answers.

(a) $\mathbb{R}$.

(b) $\mathbb{N} \times \mathbb{Q}$.

(c) The power set $\mathcal{P}(\mathbb{N})$ (=the set of all subsets of $\mathbb{N}$).

(d) The set of all **finite** subsets of $\mathbb{N}$.

Problem 2. A subset $A \subseteq \mathbb{R}$ is **open** if for every $a \in A$ there is an $\epsilon > 0$ such that the open interval $(a - \epsilon, a + \epsilon)$ is contained in A.

(a) Explain why $(0, \infty)$ is open, but $[0, \infty)$ is not.

(b) Suppose $A, B \subseteq \mathbb{R}$ are two open subsets. Show that $A \cap B$ is open.

(c) Suppose we have whole sequence of open subsets of $\mathbb{R}$, say

$$A_1, A_2, A_3, \ldots, A_n, \ldots.$$

Can one conclude that their intersection $\bigcap_{n=1}^{\infty} A_n$ is also open? Prove it in general or give a counterexample.

Problem 3. For each of the functions below indicate whether it is injective, and whether it is surjective. If a function is bijective give its inverse function.

(a) $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = x^2$.

(b) $g : \mathbb{R} \longrightarrow [0, \infty)$ given by $g(x) = 2x^2$.

(c) $h : [0, \infty) \longrightarrow \mathbb{R}$ given by $h(x) = 3x^2$.

(d) $k : [0, \infty) \longrightarrow [0, \infty)$ given by $k(x) = 4x^2$.

Problem 4. Fix a positive integer N throughout this problem.

(a) Suppose we are given more than N numbers in $[0,1]$. Explain why at least two of them must be within a distance $\frac{1}{N}$ of each other.

(b) Suppose we are given more than N^2 points in the square $[0,1] \times [0,1]$. Explain why at least two of the points must be within a distance $\frac{\sqrt{2}}{N}$.

Problem 5. Define a function $f : \{1, 2, 3, 4, 5\} \longrightarrow \{6, 7, 8, 9\}$ as follows:

$$f(1) = 7 \qquad f(2) = 6 \qquad f(3) = 9 \qquad f(4) = 7 \qquad f(5) = 9.$$

(a) List all elements of its range $f(\{1, 2, 3, 4, 5\})$.

(b) Give all elements of the two inverse images $f^{-1}(\{6, 7\})$ and $f^{-1}(\{8, 9\})$.

(c) Let $g : \{0, 1\} \to \{1, 2, 3, 4, 5\}$ be the function with values $g(0) = 3$ and $g(1) = 4$. Find the two values $(f \circ g)(0)$ and $(f \circ g)(1)$.

(d) Is $f \circ g$ injective or surjective? Does $f \circ g$ have an inverse function?

Problem 6. For an integer a we let $[a]$ denote its residue class modulo 6.

(a) List all elements of $[3]$ belonging to the open interval $(-10, 10)$.

(b) Find integers a, b in the range $0 \leq a, b < 6$ such that

$$[3] + [4] = [a] \qquad [3] \bullet [4] = [b].$$

(c) Introduce the subset $S \subseteq \mathbb{Z}$ consisting of all products xy for varying $x \in [3]$ and $y \in [4]$. Is S a residue class modulo 6?

Problem 7. Let $\mathbb{R}^\times = \mathbb{R} - \{0\}$ denote the set of nonzero real numbers. Define a relation $\sim$ on $\mathbb{R}^\times$ by declaring that, for nonzero $a, b \in \mathbb{R}$,

$$a \sim b \iff \exists r \in \mathbb{Q} : ra = b.$$

(a) Verify that $\sim$ is an equivalence relation on $\mathbb{R}^\times$.

(b) Which of the following statements are true? Explain.

  (i) $\frac{1}{\sqrt{5}} \sim \sqrt{5}$

  (ii) $\frac{\pi}{3} \sim \pi$

  (iii) $\frac{3}{5} \sim \sqrt{2}$

(c) Show that the equivalence class $[2020]$ is the same as $\mathbb{Q}^\times = \mathbb{Q} - \{0\}$.

Problem 8. Define three sequences of numbers as follows.

$$T_n = \sum_{r=1}^{n} r \qquad S_n = \sum_{r=1}^{n} r^2 \qquad R_n = \sum_{r=1}^{n} r^3.$$

(a) Use induction to verify the formula $T_n = \frac{1}{2}n(n+1)$ for all $n \in \mathbb{N}$.

(b) Use induction to verify the formula $S_n = \frac{1}{6}n(n+1)(2n+1)$ for all $n \in \mathbb{N}$.

(c) Calculate R_n for $n = 1, 2, 3, 4, 5$. Guess an explicit formula for R_n in terms of n, and prove it by induction. (`Hint`: List T_n for $n = 1, 2, 3, 4, 5$.)

Problem 9. A sequence of real numbers $x_1, x_2, x_3, \ldots$ is said to be **Cauchy** if $\forall \epsilon > 0$ there is an $N \in \mathbb{N}$ such that $|x_m - x_n| < \epsilon$ for any two indices $m, n \geq N$.

(a) Prove that any convergent sequence is Cauchy.

(b) Prove that any Cauchy sequence is bounded[1].

(c) Give an example of a bounded sequence which is <u>not</u> Cauchy.

[1] This means all its terms x_n lie in some interval of finite length.

Extra page I.

Extra page II.

Extra page III.