Partial Solutions for HW0

Problem C.
(a) Applying the distribution law, we have

\[(x - 1)(x + 2) = x^2 + (-1 + 2)x + (-1) \cdot 2 = x^2 + x - 2.\]

(b) Let \( f(x) := x^2 - 2x - 3 \). Note that

\[f(-1) = (-1)^2 - 2(-1) - 3 = 1 + 2 - 3 = 0.\]

Therefore \( f(x) \) will have \( (x + 1) \) as a factor, i.e., we can take \( \alpha = -1 \). One the other hand, we can check by computation that

\[2 = \alpha + \beta,\]

hence \( \beta = 3 \).

Problem D.
(a) Let \( x, y \) be two odd numbers. By definition of being an odd number, we have

\[x = 2n + 1, \quad y = 2m + 1\]

for some \( n, m \in \mathbb{Z} \). Now we add these two odd numbers

\[x + y = (2n + 1) + (2m + 1) = 2n + 2m + 2 = 2(n + m + 1).\]

Since \( n + m + 1 \) is some integer, \( x + y \) is an even number by definition of being an even number.

(b) We take \( x, y \) similarly as in (a). Now we take a product of these

\[x \cdot y = (2n + 1)(2m + 1) = 4nm + 2n + 2m + 1 = 2(2nm + n + m) + 1.\]

Since \( 2nm + n + m \) is some integer, \( x \cdot y \) is an even number by definition.

(c) We prove by contradiction. Let \( x, y \) be two integers. Suppose that \( x \cdot y \) is even number but both of \( x \) and \( y \) are odd numbers. By part (b), the product of odd numbers is odd which contradicts the assumption. This completes the proof.

Problem F. Define

\[T_n := 1 + 2 + 3 + \cdots + n = \sum_{k=1}^{n} k.\]
We use a standard trick to find the summatory function of arithmetic progression.

\[
T_n = 1 + 2 + 3 + \cdots + n
\]

\[
T_n = n + (n - 1) + (n - 2) + \cdots + 1
\]

By adding up two rows, we obtain

\[
2T_n = (n + 1) + (n + 1) + (n + 1) + \cdots + (n + 1) = n(n + 1).
\]

Therefore, we conclude

\[
T_n = \frac{n(n + 1)}{2}.
\]

In particular,

\[
T_{1000} = \frac{1000 \times 10001}{2} = 5000500.
\]

**Problem G.**

(c) Straightforward calculation gives

\[
\sum_{i=1}^{5} i + i^3 = (1 + 1^3) + (2 + 2^3) + \cdots + (5 + 5^3)
\]

\[
= (1 + 1) + (2 + 8) + (3 + 27) + (4 + 64) + (5 + 125)
\]

\[
= 240.
\]