Partial Solutions for HW0

Problem C.

(a) Applying the distribution law, we have

$$(x-1)(x+2) = x^2 + (-1+2)x + (-1)\cdot 2 = x^2 + x - 2.$$

(b) Let $f(x) := x^2 - 2x - 3$. Note that

$$f(-1) = (-1)^2 - 2(-1) - 3 = 1 + 2 - 3 = 0.$$

Therefore $f(x)$ will have $(x+1)$ as a factor, i.e., we can take $\alpha = -1$. One the other hand, we can check by computation that

$$2 = \alpha + \beta,$$

hence $\beta = 3$.

Problem D.

(a) Let x, y be two odd numbers. By definition of being an odd number, we have

$$x = 2n+1, \quad y = 2m+1$$

for some $n, m \in \mathbb{Z}$. Now we add these two odd numbers

$$\begin{aligned} x + y &= (2n+1) + (2m+1) \\ &= 2n + 2m + 2 \\ &= 2(n+m+1). \end{aligned}$$

Since $n+m+1$ is some integer, $x+y$ is an even number by definition of being an even number.

(b) We take x, y similarly as in (a). Now we take a product of these

$$\begin{aligned} x \cdot y &= (2n+1)(2m+1) \\ &= 4nm + 2n + 2m + 1 \\ &= 2(2nm + n + m) + 1. \end{aligned}$$

Since $2nm + n + m$ is some integer, $x \cdot y$ is an even number by definition.

(c) We prove by contradiction. Let x, y be two integers. Suppose that $x \cdot y$ is even number but both of x and y are odd numbers. By part (b), the product of odd numbers is odd which contradicts the assumption. This completes the proof.

Problem F. Define

$$T_n := 1 + 2 + 3 + \cdots + n = \sum_{k=1}^{n} k.$$

We use a standard trick to find the summatory function of arithmetic progression.

$$\begin{aligned} T_n &= 1 + \quad 2 \quad + \quad 3 \quad + \cdots + n \\ T_n &= n + (n-1) + (n-2) + \cdots + 1 \end{aligned}$$

By adding up two rows, we obtain

$$2T_n = (n+1) + (n+1) + (n+1) + \cdots + (n+1) = n(n+1).$$

Therefore, we conclude

$$T_n = \frac{n(n+1)}{2}.$$

In particular,

$$T_{1000} = \frac{1000 \times 10001}{2} = 5000500.$$

Problem G.
(c) Straightforward calculation gives

$$\begin{aligned} \sum_{i=1}^{5} i + i^3 &= (1 + 1^3) + (2 + 2^3) + \cdots + (5 + 5^3) \\ &= (1+1) + (2+8) + (3+27) + (4+64) + (5+125) \\ &= 240. \end{aligned}$$