Partial Solutions for HW1

Exercise 1.8

(a) $A = \{-3, -2, 2, 3\}$

(b) 2.1, π, or $\frac{7}{3}$.

(c) For $x \in \mathbb{R}$,

$$x^2 - (2+\sqrt{2})x + 2\sqrt{2} = (x-2)(x-\sqrt{2}) = 0 \quad \text{if and only if} \quad x = 2 \text{ or } x = \sqrt{2},$$

we have $C = \{2, \sqrt{2}\}$.

(d) For $x \in \mathbb{Q}$,

$$x^2 - (2+\sqrt{2})x + 2\sqrt{2} = 0 \quad \text{if and only if} \quad x = 2,$$

because $\sqrt{2} \notin \mathbb{Q}$. Therefore $D = \{2\}$.

(e) $|A| = 4$, $|C| = 2$, and $|D| = 1$.

Exercise 1.14 Fine $\mathcal{P}(A)$ and $|\mathcal{P}(A)|$ for following A's.

(a) Let $A = \{1, 2\}$. Then the power set of A is

$$\mathcal{P}(A) = \big\{\emptyset, \{1\}, \{2\}, \{1, 2\}\big\}$$

and $|\mathcal{P}(A)| = 4$ as expected from the formula

$$|\mathcal{P}(A)| = 2^{|A|} = 2^2 = 4.$$

(b) Let $A = \{\emptyset, 1, \{a\}\}$. Then the power set of A is

$$\mathcal{P}(A) = \Big\{\emptyset, \{\emptyset\}, \{1\}, \{\{a\}\}, \{1, \{a\}\}, \{\{a\}, \emptyset\}, \{\emptyset, 1\}, \{\emptyset, 1, \{a\}\}\Big\}$$

and $|\mathcal{P}(A)| = 8$ as expected from the formula

$$|\mathcal{P}(A)| = 2^{|A|} = 2^3 = 8.$$

Exercise 1.20

(a) False. Let $A = \{1, \{1\}\}$. This is a counterexample of the statement because even if $\{1\} \in \mathcal{P}(A)$, we have $1 \in A$ and $\{1\} \in A$.

(b) True. Suppose $A \subset \mathcal{P}(B) \subset C$ and $|A| = 2$. Then we have a strict inequalities (because $\subset$ only means proper subset) of the cardinalities

$$2 = |A| < 2^{|B|} < |C|.$$

This implies that $|B| \geq 4$ and so $|C| > 4$. This proves that $|C|$ cannot be 4. On the other hand, $|C|$ can be 5 if we choose

$$A = \{\emptyset, \{1\}\}, \quad B = \{1, 2\}, \quad C = \big\{\emptyset, \{1\}, \{2\}, \{1, 2\}, 3\big\}.$$

(c) False. Let $A = \emptyset$ and $B = \{1\}$. This is a counterexample of the statement because even if $|B| = 1 = |A| + 1$, we have $|\mathcal{P}(B)| = 2$ and $|\mathcal{P}(A)| = 1$ which differ by 1.

(d) True. Note that subsets of $\{1, 2, 3\}$ of cardinality 2 are exactly one of the followings:

$$\{1, 2\}, \quad \{2, 3\}, \quad \{1, 3\}.$$

Since there are only three of them, if A, B, C, and D are subsets of $\{1, 2, 3\}$ of cardinality 2, then there should be at least one repetition. This proves the claim.

Exercise 1.24 Let $A = \{1, 2\}$, $B = \{1, 2\}$, and $C = \{1\}$. Then

$$B \neq C \quad \text{but} \quad B - A = \emptyset = C - A.$$

Exercise 1.46

(a) (i) We claim that

$$\bigcup_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = (-1, 1).$$

To prove that two sets are equal, we need to show that one includes the other and vice versa. First, we show that

$$\bigcup_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) \subseteq (-1, 1).$$

Let $x \in \bigcup_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right)$. By definition of union of indexed sets, we have

$$x \in \left(-\frac{1}{n}, \frac{1}{n}\right) \quad \text{for some} \quad n \geq 1.$$

Therefore, we have

$$-1 \leq -\frac{1}{n} < x < \frac{1}{n} \leq 1.$$

In particular, $x \in (-1, 1)$. Now we show that

$$\bigcup_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) \supseteq (-1, 1).$$

Let $x \in (-1, 1)$. Then it satisfies $x \in \left(-\frac{1}{n}, \frac{1}{n}\right)$ for $n = 1$. This, by definition, implies that $x \in \bigcup_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right)$.

(ii) We claim that

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}.$$

First, we show $\subseteq$ direction. Let $x \in \bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right)$. By definition of intersection of indexed sets, we have

$$x \in \left(-\frac{1}{n}, \frac{1}{n}\right) \quad \text{for all} \quad n \geq 1.$$

In other words, $|x| < \frac{1}{n}$ for all $n \geq 1$. Therefore, we have

$$|x| \leq \lim_{n\to\infty} \frac{1}{n} = 0$$

which implies $x = 0$. Note that we have changed strict inequality to inclusive inequality when we take limit. This proves $\subseteq$ direction. Conversely, for $x = 0$, it is clear that

$$0 \in \left(-\frac{1}{n}, \frac{1}{n}\right) \quad \text{for all} \quad n \geq 1.$$

This proves $\subseteq$ direction, hence the claim follows.

(b) (i) We claim that

$$\bigcup_{n=1}^{\infty} \left[\frac{n-1}{n}, \frac{n+1}{n}\right] = [0, 2]$$

We first show $\subseteq$ direction. Let $x \in \bigcup_{n=1}^{\infty} \left[\frac{n-1}{n}, \frac{n+1}{n}\right]$. By definition,

$$x \in \left[\frac{n-1}{n}, \frac{n+1}{n}\right] \quad \text{for some} \quad n \geq 1.$$

Therefore, we have

$$0 \leq 1 - \frac{1}{n} = \frac{n-1}{n} \leq x \leq \frac{n+1}{n} = 1 + \frac{1}{n} \leq 2.$$

In particular, $x \in [0, 2]$. Now we show $\supseteq$ direction. Let $x \in [0, 2]$. Then it satisfies $x \in \left[\frac{n-1}{n}, \frac{n+1}{n}\right]$ for $n = 1$. This implies that $x \in \bigcup_{n=1}^{\infty} \left[\frac{n-1}{n}, \frac{n+1}{n}\right]$. This proves the claim.

(ii) We claim that

$$\bigcap_{n=1}^{\infty} \left[\frac{n-1}{n}, \frac{n+1}{n}\right] = \{1\}.$$

First, we show $\subseteq$ direction. Let $x \in \bigcap_{n=1}^{\infty} \left[\frac{n-1}{n}, \frac{n+1}{n}\right]$. By definition of intersection of indexed sets, we have

$$x \in \left[\frac{n-1}{n}, \frac{n+1}{n}\right] \quad \text{for all} \quad n \geq 1.$$

In other words, $|x - 1| \leq \frac{1}{n}$ for all $n \geq 1$. Therefore, we have

$$|x - 1| \leq \lim_{n\to\infty} \frac{1}{n} = 0$$

which implies $x = 0$. This proves $\subseteq$ direction. Conversely, for $x = 1$, it is clear that

$$1 \in \left[\frac{n-1}{n}, \frac{n+1}{n}\right] = \left[1 - \frac{1}{n}, 1 + \frac{1}{n}\right] \quad \text{for all} \quad n \geq 1.$$

This proves $\subseteq$ direction, hence the claim follows.