Partial Solutions for HW1

Exercise 1.8

(a) \( A = \{-3, -2, 2, 3\} \)

(b) \(2, 1, \pi\), or \(\frac{7}{5}\).

(c) For \(x \in \mathbb{R}\),
\[
x^2 - (2 + \sqrt{2})x + 2\sqrt{2} = (x - 2)(x - \sqrt{2}) = 0 \quad \text{if and only if} \quad x = 2 \text{ or } x = \sqrt{2},
\]
we have \(C = \{2, \sqrt{2}\}\).

(d) For \(x \in \mathbb{Q}\),
\[
x^2 - (2 + \sqrt{2})x + 2\sqrt{2} = 0 \quad \text{if and only if} \quad x = 2,
\]
because \(\sqrt{2} \notin \mathbb{Q}\). Therefore \(D = \{2\}\).

(e) \(|A| = 4, |C| = 2, \text{ and } |D| = 1|\).

Exercise 1.14

Fine \(\mathcal{P}(A)\) and \(|\mathcal{P}(A)|\) for following \(A\)'s.

(a) Let \(A = \{1, 2\}\). Then the power set of \(A\) is
\[
\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}
\]
and \(|\mathcal{P}(A)| = 4\) as expected from the formula
\[
|\mathcal{P}(A)| = 2^{|A|} = 2^2 = 4.
\]

(b) Let \(A = \{\emptyset, 1, \{a\}\}\). Then the power set of \(A\) is
\[
\mathcal{P}(A) = \{\emptyset, \emptyset, \{1\}, \{\{a\}\}, \{1, \{a\}\}, \{\{a\}, \emptyset\}, \emptyset, 1, \{\emptyset, 1, \{a\}\}\}
\]
and \(|\mathcal{P}(A)| = 8\) as expected from the formula
\[
|\mathcal{P}(A)| = 2^{|A|} = 2^3 = 8.
\]

Exercise 1.20

(a) False. Let \(A = \{1, \{1\}\}\). This is a counterexample of the statement because even if \(\{1\} \in \mathcal{P}(A)\), we have \(1 \in A\) and \(\{1\} \in A\).

(b) True. Suppose \(A \subseteq \mathcal{P}(B) \subseteq C\) and \(|A| = 2\). Then we have a strict inequalities (because \(\subseteq\) only means proper subset) of the cardinalities
\[
2 = |A| < 2^{|B|} < |C|.
\]
This implies that \(|B| \geq 4\) and so \(|C| > 4\). This proves that \(|C|\) cannot be 4. On the other hand, \(|C|\) can be 5 if we choose
\[
A = \{\emptyset, \{1\}\}, \quad B = \{1, 2\}, \quad C = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, 3\}.
\]
(c) False. Let \( A = \emptyset \) and \( B = \{1\} \). This is a counterexample of the statement because even if \( |B| = 1 = |A| + 1 \), we have \( |\mathcal{P}(B)| = 2 \) and \( |\mathcal{P}(A)| = 1 \) which differ by 1.

(d) True. Note that subsets of \( \{1, 2, 3\} \) of cardinality 2 are exactly one of the followings:

\[ \{1, 2\}, \{2, 3\}, \{1, 3\}. \]

Since there are only three of them, if \( A, B, C, \) and \( D \) are subsets of \( \{1, 2, 3\} \) of cardinality 2, then there should be at least one repetition. This proves the claim.

**Exercise 1.24** Let \( A = \{1, 2\} \), \( B = \{1, 2\} \), and \( C = \{1\} \). Then

\[ B \neq C \quad \text{but} \quad B - A = \emptyset = C - A. \]

**Exercise 1.46**

(a) (i) We claim that

\[ \bigcup_{n=1}^{\infty} \left( -\frac{1}{n}, \frac{1}{n} \right) = (-1, 1). \]

To prove that two sets are equal, we need to show that one includes the other and vice versa. First, we show that

\[ \bigcup_{n=1}^{\infty} \left( -\frac{1}{n}, \frac{1}{n} \right) \subseteq (-1, 1). \]

Let \( x \in \bigcup_{n=1}^{\infty} \left( -\frac{1}{n}, \frac{1}{n} \right) \). By definition of union of indexed sets, we have

\[ x \in \left( -\frac{1}{n}, \frac{1}{n} \right) \quad \text{for some} \quad n \geq 1. \]

Therefore, we have

\[ -1 \leq -\frac{1}{n} < x < \frac{1}{n} \leq 1. \]

In particular, \( x \in (-1, 1) \). Now we show that

\[ \bigcup_{n=1}^{\infty} \left( -\frac{1}{n}, \frac{1}{n} \right) \supseteq (-1, 1). \]

Let \( x \in (-1, 1) \). Then it satisfies \( x \in \left( -\frac{1}{n}, \frac{1}{n} \right) \) for \( n = 1 \). This, by definition, implies that \( x \in \bigcup_{n=1}^{\infty} \left( -\frac{1}{n}, \frac{1}{n} \right) \).

(ii) We claim that

\[ \bigcap_{n=1}^{\infty} \left( -\frac{1}{n}, \frac{1}{n} \right) = \{0\}. \]

First, we show \( \subseteq \) direction. Let \( x \in \bigcap_{n=1}^{\infty} \left( -\frac{1}{n}, \frac{1}{n} \right) \). By definition of intersection of indexed sets, we have

\[ x \in \left( -\frac{1}{n}, \frac{1}{n} \right) \quad \text{for all} \quad n \geq 1. \]
In other words, $|x| < \frac{1}{n}$ for all $n \geq 1$. Therefore, we have

$$|x| \leq \lim_{n \to \infty} \frac{1}{n} = 0$$

which implies $x = 0$. Note that we have changed strict inequality to inclusive inequality when we take limit. This proves $\subseteq$ direction. Conversely, for $x = 0$, it is clear that

$$0 \in \left( -\frac{1}{n}, \frac{1}{n} \right) \text{ for all } n \geq 1.$$ 

This proves $\subseteq$ direction, hence the claim follows.

(b) (i) We claim that

$$\bigcup_{n=1}^{\infty} \left[ \frac{n-1}{n}, \frac{n+1}{n} \right] = [0, 2]$$

We first show $\subseteq$ direction. Let $x \in \bigcup_{n=1}^{\infty} \left[ \frac{n-1}{n}, \frac{n+1}{n} \right]$. By definition,

$$x \in \left[ \frac{n-1}{n}, \frac{n+1}{n} \right] \text{ for some } n \geq 1.$$ 

Therefore, we have

$$0 \leq 1 - \frac{1}{n} = \frac{n-1}{n} \leq x \leq \frac{n+1}{n} = 1 + \frac{1}{n} \leq 2.$$ 

In particular, $x \in [0, 2]$. Now we show $\supseteq$ direction. Let $x \in [0, 2]$. Then it satisfies $x \in \left[ \frac{n-1}{n}, \frac{n+1}{n} \right]$ for $n = 1$. This implies that $x \in \bigcup_{n=1}^{\infty} \left[ \frac{n-1}{n}, \frac{n+1}{n} \right]$. This proves the claim.

(ii) We claim that

$$\bigcap_{n=1}^{\infty} \left[ \frac{n-1}{n}, \frac{n+1}{n} \right] = \{1\}.$$ 

First, we show $\subseteq$ direction. Let $x \in \bigcap_{n=1}^{\infty} \left[ \frac{n-1}{n}, \frac{n+1}{n} \right]$. By definition of intersection of indexed sets, we have

$$x \in \left[ \frac{n-1}{n}, \frac{n+1}{n} \right] \text{ for all } n \geq 1.$$ 

In other words, $|x - 1| \leq \frac{1}{n}$ for all $n \geq 1$. Therefore, we have

$$|x - 1| \leq \lim_{n \to \infty} \frac{1}{n} = 0$$

which implies $x = 0$. This proves $\subseteq$ direction. Conversely, for $x = 1$, it is clear that

$$1 \in \left[ \frac{n-1}{n}, \frac{n+1}{n} \right] = \left[ 1 - \frac{1}{n}, 1 + \frac{1}{n} \right] \text{ for all } n \geq 1.$$ 

This proves $\subseteq$ direction, hence the claim follows.