Partial Solutions for HW2

Exercise 1.52 Consider the following subsets of $\mathbb{N}$:

$$A_1 = \{3n : n \in \mathbb{N}\}, \quad A_2 = \{3n+1 : n \in \mathbb{N}\}, \quad A_3 = \{3n+2 : n \in \mathbb{N}\}.$$

They are nonempty collection of subsets of $\mathbb{N}$. They partition $\mathbb{N}$ because if we divide any natural number $k \in \mathbb{N}$ by 3 then its remainder is either $0, 1,$ or 2. Therefore

$$S = \{A_1, A_2, A_3\}$$

is an example of a partition of $\mathbb{N}$ by three subsets.

Exercise 1.72
We claim that

$$\big\{A \times A,\ A \times B,\ B \times A,\ B \times B\big\} \tag{1}$$

is a partition of $S \times S$ by 4 subsets. This will follow from the fact that $\{A, B\}$ partitions S.

1. nonempty: It is clear that they are all nonempty because

   $$|A \times A| = |A| \cdot |A| = 4, \quad |A \times B| = |B \times A| = |A| \cdot |B| = 4, \quad |B \times B| = |B| \cdot |B| = 4.$$

2. Pairwise disjoint: Take any two elements in (1), say X and Y. We want to show that either $X = Y$ or $X \cap Y = \emptyset$. Equivalently, we will show that

   $$\text{if } X \cap Y \neq \emptyset, \quad \text{then} \quad X = Y.$$

   Suppose $X \cap Y \neq \emptyset$. Pick any $(p, q) \in X \cap Y$. Since $p \in S$ and S is partitioned by A and B, p belongs to exactly one of A or B. Similarly, q belongs to exactly one of A or B. There are 4 possibilities here. As an example, let's say $p \in A$ and $q \in B$. Then only possibility for X and Y will be $A \times B$, in particular, $X = Y$. In general, position of p and q in terms of A and B will determine the subset X and Y of $S \times S$. Therefore $X = Y$.

3. Union is everything: Now we will show that

   $$S \times S = \big(A \times A\big) \cup \big(A \times B\big) \cup \big(B \times A\big) \cup \big(B \times B\big).$$

   This means that for any $(p, q) \in S \times S$, it belongs to at least one element in (1). Since $p, q \in S$, each of them belong to at least one of A or B. (In fact, they belong to exactly one of A or B, but we won't need this for now). As before, there are 4 possibilities. As an example, let's say that $p \in B$ and $q \in B$. Then $(p, q) \in B \times B$. In any other cases, one can show that (p, q) belongs to at least one of $A \times A,\ A \times B,\ B \times A,\ B \times B$. This completes the proof.

Exercise 4.46 Prove that $A \cup B = A \cap B$ if and only if $A = B$.

$\Longleftarrow$: Suppose $A = B$. Then it is clear that $A \cup B = A = A \cap B$.

$\Longrightarrow$: Suppose $A \cup B = A \cap B$. Then we have

$$A \subseteq A \cup B = A \cap B \subseteq B.$$

Similarly,

$$B \subseteq A \cup B = A \cap B \subseteq A.$$

Therefore $A = B$.

Exercise 4.56 Prove that $(A - B) \cup (A - C) = A - (B \cap C)$.

Proof.

$$\begin{aligned}
x \in (A-B) \cup (A-C) &\iff x \in (A-B) \text{ or } x \in (A-C) \\
&\iff \Big[x \in A \text{ and } x \notin B\Big] \text{ or } \Big[x \in A \text{ and } x \notin C\Big] \\
&\iff x \in A \text{ and } \Big[x \notin B \text{ or } x \notin C\Big] \\
&\iff x \in A \text{ and } \sim \Big[x \in B \text{ and } x \in C\Big] \\
&\iff x \in A \text{ and } \sim \Big[x \in B \cap C\Big] \\
&\iff x \in A \text{ and } x \notin B \cap C \\
&\iff x \in A - (B \cap C).
\end{aligned}$$

$\square$

Exercise 4.68 Prove that $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

Proof.

$$\begin{aligned}
(x,y) \in (A \times B) \cap (C \times D) &\iff (x,y) \in A \times B \text{ and } (x,y) \in C \times D \\
&\iff \Big[x \in A \text{ and } y \in B\Big] \text{ and } \Big[x \in C \text{ and } y \in D\Big] \\
&\iff \Big[x \in A \text{ and } x \in C\Big] \text{ and } \Big[y \in B \text{ and } y \in D\Big] \\
&\iff x \in A \cap C \text{ and } y \in B \cap D \\
&\iff (x,y) \in (A \cap C) \times (B \cap D).
\end{aligned}$$

$\square$