Partial Solutions for HW3

Exercise 2.20 For statements P and Q, construct a truth table for $(P \Rightarrow Q) \Rightarrow (\sim Q)$.

P	Q	$P \Rightarrow Q$	$(\sim Q)$	$(P \Rightarrow Q) \Rightarrow (\sim Q)$
T	T	T	F	F
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

Exercise 2.32 We will give an uniform treatment for each case as follows. Define

$$A := \{x \in S \,|\, P(x) \text{ is true }\}$$
$$B := \{x \in S \,|\, Q(x) \text{ is true }\}.$$

Since

$$P(x) \Rightarrow Q(x) \;\equiv\; (\sim P(x)) \vee (P(x) \wedge Q(x)),$$

the set of all $x \in S$ for which $P(x) \Rightarrow Q(x)$ is true is given by

$$\{x \in S \,|\, P(x) \Rightarrow Q(x) \text{ is true}\} = \overline{A} \cup (A \cap B).$$

(a) $P(x) : x - 3 = 4;\; Q(x) : x \geq 8;\; S = \mathbb{R}$.
We have

$$A = \{7\}, \quad B = [8, \infty).$$

Therefore

$$\begin{aligned} \overline{A} \cup (A \cap B) &= \overline{\{7\}} \cup (\{7\} \cap [8, \infty)) \\ &= \{x \in \mathbb{R} \,|\, x \neq 7\} \cup \emptyset \\ &= \{x \in \mathbb{R} \,|\, x \neq 7\}. \end{aligned}$$

(b) $P(x) : x^2 \geq 1;\; Q(x) : x \geq 1;\; S = \mathbb{R}$.
We have

$$A = \{x \in \mathbb{R} \,|\, x^2 \geq 1\} = (-\infty, -1] \cup [1, \infty), \quad B = [1, \infty).$$

Therefore

$$\begin{aligned} \overline{A} \cup (A \cap B) &= (-1, 1) \cup \Big(\big((-\infty, -1] \cup [1, \infty)\big) \cap [1, \infty)\Big) \\ &= (-1, 1) \cup [1, \infty) \\ &= (-1, \infty) \\ &= \{x \in \mathbb{R} \,|\, x > -1\}. \end{aligned}$$

(c) $P(x) : x^2 \geq 1;\; Q(x) : x \geq 1;\; S = \mathbb{N}$.
We have

$$A = \{x \in \mathbb{N} \,|\, x^2 \geq 1\} = \mathbb{N}, \quad B = \{x \in \mathbb{N} \,|\, x \geq 1\} = \mathbb{N}.$$

Therefore

$$\begin{aligned} \overline{A} \cup (A \cap B) &= \emptyset \cup (\mathbb{N} \cap \mathbb{N}) \\ &= \mathbb{N}. \end{aligned}$$

(d) $P(x) : x \in [-1,2]$; $Q(x) : x^2 \leq 2$; $S = [-1,1]$. We have

$$A = \{x \in [-1,1] \,|\, x \in [-1,2]\} = S, \quad B = \{x \in [-1,1] \,|\, x^2 \leq 2\} = S.$$

This is because for any $x \in S = [-1,1]$, it satisfies both $x \in [-1,2]$ and $x^2 \leq 1 \leq 2$. Therefore

$$\begin{aligned}\overline{A} \cup (A \cap B) &= \emptyset \cup (S \cap S) \\ &= S \\ &= [-1,1].\end{aligned}$$

Exercise 2.56 We want to show that $(\sim Q) \Rightarrow (P \wedge (\sim P))$ and Q are logically equivalent. We do this by drawing the truth table.

P	Q	$(\sim P)$	$(P \wedge (\sim P))$	$(\sim Q)$	$(\sim Q) \Rightarrow (P \wedge (\sim P))$
T	T	F	F	F	T
T	F	F	F	T	F
F	T	T	F	F	T
F	F	T	F	T	F

Since truth value of $(\sim Q) \Rightarrow (P \wedge (\sim P))$ and Q are identical no matter what truth value of P and Q are, they are logically identical.

Exercise 2.72

(a) True. Take $x = 0 \in \mathbb{R}$. It clearly satisfies $x^2 - x = 0$.

(b) True. For any $n \in \mathbb{N}$, we have $n \geq 1$ hence $n + 1 \geq 1 + 1 = 2$.

(c) False. Take $x = -1 \in \mathbb{R}$. Then $\sqrt{(-1)^2} = \sqrt{1} = 1 \neq -1$.

(d) True. Take $x = 3 \in \mathbb{Q}$. It clearly satisfies $3x^2 - 27 = 0$.

(e) True. Take $x = 5 \in \mathbb{R}$ and $y = 0 \in \mathbb{R}$. They clearly satisfies $x + y + 3 = 8$.

(f) False. Take $x = y = 0 \in \mathbb{R}$. Then $x + y + 3 = 3 \neq 8$.

(g) True. Take $x = 3 \in \mathbb{R}$ and $y = 0 \in \mathbb{R}$. They clearly satisfies $x^2 + y^2 = 9$.

(h) False. Take $x = y = 0 \in \mathbb{R}$. Then $x^2 + y^2 = 0 \neq 9$.

Exercise 2.78

(a) For any circle C_1 in the plane with center $(0,0)$, there exists a circle C_2 in the plane with center $(1,1)$ such that C_1 and C_2 have exactly two points in common.

(b)

$$\exists C_1 \in A, \ \forall C_2 \in B, \ \sim P(C_1, C_2).$$

(c) There exists a circle C_1 in the plane with center $(0,0)$ such that for any circle C_2 in the plane with center $(1,1)$, C_1 and C_2 do not have exactly two points in common.

Alternative answer: There exists a circle in the plane with center $(0,0)$ with which no circle in the plane with center $(1,1)$ intersect have exactly two points in common.