Partial Solutions for HW4

Exercise 14.4 Prove that the sequence $\left\{\frac{1}{n^2+1}\right\}$ converges to 0.

Proof. Let $\epsilon > 0$.

$$\underline{\text{Want}} : N \in \mathbb{N} \text{ such that if } n \geq N \text{ then } \left|\frac{1}{n^2+1} - 0\right| < \epsilon.$$

We analyze the desired inequality as follows:

$$\begin{aligned}\left|\frac{1}{n^2+1} - 0\right| < \epsilon &\iff n^2 + 1 > \frac{1}{\epsilon}\\ &\iff n^2 > \frac{1}{\epsilon} - 1\\ &\Longleftarrow \ n > \sqrt{\left|\frac{1}{\epsilon} - 1\right|}.\end{aligned}$$

Note that the last line is not "iff" in general since $\frac{1}{\epsilon} - 1$ can be negative. In any case, we only need one direction which implies the desired inequality. Now we just pick any big enough natural number N which satisfies

$$N > \sqrt{\left|\frac{1}{\epsilon} - 1\right|}.$$

We can find such N because ϵ is now fixed. This completes the proof.

□

Exercise 9.18 Let $A = \{1, 2, 3, 4\}$. Give an example of a relation on A that is:

(a) reflexive and symmetric but not transitive.
Define
$$R := \{(1,1), (2,2), (3,3), (4,4), (1,2), (2,3), (2,1), (3,2)\}.$$
It is clearly reflexive and symmetric. But it is not transitive because $(1,2), (2,3) \in R$ but $(1,3) \notin R$.

(c) symmetric and transitive but not reflexive.
Define
$$R := \{(1,1)\}$$
It is clearly symmetric and transitive. But it is not reflexive because $(2,2) \notin R$.

(e) symmetric but neither reflexive nor transitive. Define
$$R := \{(1,2), (2,1)\}.$$
It is clearly symmetric. But it is not reflexive because $(1,1) \notin R$. Also, it is not transitive because $(1,2), (2,1) \in R$ but $(1,1) \notin R$.

Exercise 9.34 Let H be a nonempty subset of $\mathbb{Z}$. Suppose that the relation on R defined on Z by $a\,R\,b$ if $a-b\in H$ is an equivalence relation. Verify the following:

(a) $0 \in H$.

Proof. Let $n\in\mathbb{Z}$. By reflexive property, we have $n\,R\,n$, i.e., $0 = n-n\in H$. □

(b) If $a\in H$, then $-a\in H$.

Proof. Suppose $a\in H$. Note that $a\,R\,0$ because $a-0=a\in H$. By symmetric property, we also have $0\,R\,a$, i.e., $-a = 0-a\in H$. □

(c) If $a,b\in H$, then $a+b\in H$.

Proof. Suppose $a,\ b\in H$. We claim that

$$(a+b)\,R\,b \quad \text{and} \quad b\,R\,0.$$

First claim is because $(a+b)-b = a\in H$. Second claim is because $b-0=b\in H$. By transitive property, we have $(a+b)\,R\,0$, i.e., $a+b = (a+b)-0\in H$. □

Exercise 9.46 Let R be the relation defined on $\mathbb{Z}$ by $a\ R\ b$ if $a+b\equiv 0 \pmod 3$. Show that R is not an equivalence relation.

Proof. 1) It is not reflexive. Consider $a=b=1$. Then $a+b\equiv 2 \pmod 3$. This is not equivalent to 0, hence $(1,1)\notin R$.
2) It is symmetric.
3) It is not transitive. Consider $a=1$, $b=2$, $c=1$. Then $a+b\equiv b+c\equiv 0 \pmod 3$, but $a+c\equiv 2 \pmod 3$. This is not equivalent to 0. Therefore $(a,b),(b,c)\in R$ but $(a,c)\notin R$. □

Exercise 9.58 Prove that the multiplication in $\mathbb{Z}_n$, $n\geq 2$, defined by $[a][b]=[ab]$ is well-defined.

Proof. We need to show that resulting residue class $[ab]$ does not depend on the choice of representatives of $[a]$ and $[b]$. To this end, suppose

$$[a]=[a'] \quad \text{and} \quad [b]=[b'].$$

We want to show that

$$[ab]=[a'b'].$$

By the assumption, n divides $a'-a$ and $b'-b$. Therefore there exist some integers $p,q\in\mathbb{Z}$ such that

$$a'-a=np \quad \text{and} \quad b'-b=nq.$$

We have

$$\begin{aligned} a'b'-ab &= (a+np)(b+nq)-ab \\ &= n(pb+aq+npq). \end{aligned}$$

Therefore n divides $a'b'-ab$, hence $[ab]=[a'b']$. □