# Partial Solutions for HW5

**Exercise 9.52** Let $R$ be the relation defined on $\mathbb{Z}$ by $a\,R\,b$ if $a^2 \equiv b^2 \pmod 5$. Prove that $R$ is an equivalence relation and determine the distinct equivalence classes.

*Proof.*

1. Reflexive: Pick any $a \in \mathbb{Z}$. Since $a^2 - a^2 = 0$ is divisible by 5, we have $a^2 \equiv a^2 \pmod 5$ hence $a\,R\,a$.
2. Symmetric: Suppose $a\,R\,b$, i.e., $a^2 \equiv b^2 \pmod 5$. By definition, there exists an integer $k \in \mathbb{Z}$ such that $a^2 - b^2 = 5k$. Since $b^2 - a^2 = 5(-k)$, we have $b^2 \equiv a^2 \pmod 5$, i.e., $b\,R\,a$.
3. Transitive: Suppose $a\,R\,b$ and $b\,R\,c$. This means that there exist integers $k_1, k_2 \in \mathbb{Z}$ such that
$$a^2 - b^2 = 5k_1 \quad \text{and} \quad b^2 - c^2 = 5k_2.$$
By adding up two equalities, we have
$$a^2 - c^2 = (a^2 - b^2) + (b^2 - c^2) = 5(k_1 + k_2).$$
Since $k_1 + k_2$ is an integer, we have $a^2 \equiv c^2 \pmod 5$ hence $a\,R\,c$.

Now we determine the distinct equivalence classes. The equivalence class including $a$ is

$$\begin{aligned}
[a] &= \{b \in \mathbb{Z} : a^2 \equiv b^2 \pmod 5\} \\
&= \{b \in \mathbb{Z} : 5 \text{ divides } b^2 - a^2 = (b-a)(b+a)\} \\
&= \{b \in \mathbb{Z} : 5|(b-a) \text{ or } 5|(b+a)\} \\
&= \{b \in \mathbb{Z} : 5|(b-a)\} \cup \{b \in \mathbb{Z} : 5|(b+a)\} \\
&= \{b \in \mathbb{Z} : b-a = 5k,\ k \in \mathbb{Z}\} \cup \{b \in \mathbb{Z} : b+a = 5k,\ k \in \mathbb{Z}\} \\
&= \{a + 5k : k \in \mathbb{Z}\} \cup \{-a + 5k : k \in \mathbb{Z}\},
\end{aligned}$$

where we used the primality of 5 in the third equality. Applying this to the case when $a = 0, 1, 2$, we obtain

$$\begin{aligned}
[0] &= \{5k : k \in \mathbb{Z}\} \\
[1] &= \{1 + 5k : k \in \mathbb{Z}\} \cup \{-1 + 5k : k \in \mathbb{Z}\} \\
&= \{1 + 5k : k \in \mathbb{Z}\} \cup \{4 + 5k : k \in \mathbb{Z}\} \\
[2] &= \{2 + 5k : k \in \mathbb{Z}\} \cup \{-2 + 5k : k \in \mathbb{Z}\} \\
&= \{2 + 5k : k \in \mathbb{Z}\} \cup \{3 + 5k : k \in \mathbb{Z}\}.
\end{aligned}$$

Note that these are all distinct equivalence classes because every integer belongs to at least one of them depending on the residue modulo 5. □

**Exercise 9.60** For integers $m, n \geq 2$ consider $\mathbb{Z}_m$ and $\mathbb{Z}_n$. Let $[a] \in \mathbb{Z}_m$ where $0 \leq a \leq m-1$. Then $a, a+m \in [a]$ in $\mathbb{Z}_m$. If $a, a+m \in [b]$ for some $[b] \in \mathbb{Z}_n$, then what can be said of $m$ and $n$?

*Solution.* Suppose that $a, a+m \in [b]$ for some $[b] \in \mathbb{Z}_n$. Note that element belongs to the same equivalence class if and only if they are related. Therefore $a$ and $a+m$ have same residue modulo $n$, i.e., $n$ divides $(a+m)-a = n$. So we conclude that $n$ divides $m$. □

**Exercise 10.4** For the given subset $A_i$ of $\mathbb{R}$ and the relation $R_i$ $(1 \leq i \leq 3)$ from $A_i$ to $\mathbb{R}$, determine whether $R_i$ is a function from $A_i$ to $\mathbb{R}$.

(a) $A_1 = \mathbb{R},\ R_1 = \{(x, y) : x \in A_1,\ y = 4x - 3\}$

This is a function. For any given element $x$ in the domain $\mathbb{R}$, we have uniquely determined element $y = 4x - 3$ in the codomain $\mathbb{R}$.

(b) $A_2 = [0, \infty),\ R_2 = \{(x, y) : x \in A_2,\ (y+2)^2 = x\}$

This is not a function. For an element $1 \in [0, \infty)$, the relation does not determine the value $y$ in the codomain $\mathbb{R}$ uniquely because both $y = -1$ and $y = -3$ satisfies

$$(y+2)^2 = 1.$$

Using the language of the relation, this can be written as $1\, R_2\, (-1)$ and $1\, R_2\, (-3)$.

(c) $A_3 = \mathbb{R},\ R_3 = \{(x, y) : x \in A_3,\ (x+y)^2 = 4\}$

This is not a function. Note that

$$(x+y)^2 = 4 \iff (x+y) = 2 \ \text{ or } \ (x+y) = -2.$$

Therefore, for each element $x$ in the domain $\mathbb{R}$, we have two different element $y$ in the codomain related to $x$, namely

$$y = -x + 2 \ \text{ or } \ y = -x - 2.$$

For example, we have $0\, R_3\, 2$ and $0\, R_3\, (-2)$.

**Exercise 10.8** Let $A = \{5, 6\}$, $B = \{5, 7, 8\}$ and $S = \{n : n \geq 3 \text{ is an odd integer.}\}$. A relation $R$ from $A \times B$ to $S$ is defined as $(a, b)\, R\, s$ if $s \mid (a+b)$. Is $R$ a function from $A \times B$ to $S$?

*Solution.* Yes, it is a function. We need to show that for each $(a, b) \in A \times B$, $(a+b)$ has an unique odd divisor greater than or equal to 3. For this problem, there are essentially no better ways to see this than the case study. One can easily check that the possible values of $(a+b)$ for each pair $(a, b) \in A \times B$ are the followings:

$$10, \quad 11, \quad 12, \quad 13, \quad 14.$$

It is clear that each of them has exactly one odd divisor greater than or equal to 3, namely $5, 11, 3, 13, 7$, respectively. □

**Exercise 10.16**

(a) Give an example of sets $A$ and $B$ such that $|B^A| = 8$.

It is known that for each finite sets $A$ and $B$, we have

$$|B^A| = |B|^{|A|}.$$

Therefore, any choice of sets $A$ and $B$ with $|A| = 3$ and $|B| = 2$ work because

$$|B^A| = |B|^{|A|} = 2^3 = 8.$$

To be more specific,

$$A = \{1, 2, 3\}, \quad B = \{4, 5\}$$

gives such an example.

(b) Give an example of an element in $B^A$ for the sets $A$ and $B$ given in (a).

By definition, element $f \in B^A$ is a function $f : A \to B$. We have a following example:

$$f(1) = 4, \quad f(2) = 5, \quad f(3) = 4$$

In the language of the relation, this is denoted by $f = \{(1, 4), (2, 5), (3, 4)\} \subseteq B \times A$.