Partial Solutions for HW6

Exercise 10.12 For a function $f : A \to B$ and subsets C and D of A and E and F of B, prove the following.

(a) $f(C \cup D) = f(C) \cup f(D)$

Proof. For $y \in B$, we have

$$\begin{aligned} y \in f(C \cup D) &\iff \exists x \in C \cup D \;\; s.t. \;\; y = f(x) \\ &\iff \Big[\exists x \in C \;\; s.t. \;\; y = f(x)\Big] \vee \Big[\exists x \in D \;\; s.t. \;\; y = f(x)\Big] \\ &\iff \Big[y \in f(C)\Big] \vee \Big[y \in f(D)\Big] \\ &\iff y \in f(C) \cup f(D). \end{aligned}$$

This completes the proof. □

(c) $f(C) - f(D) \subseteq f(C - D)$

Proof. Let $y \in f(C) - f(D)$. Then $y \in f(C)$ but $y \notin f(D)$. Since $y \in f(C)$, there is some $x \in C$ such that $y = f(x)$. On the other hand, if x were in D, then this would imply that $y \in f(D)$ which is a contradiction. Therefore x is not in D. Since $x \in C$ and $x \notin D$, we have $x \in C - D$. From $y = f(x)$, we conclude that $y \in f(C - D)$. This completes the proof. □

(e) $f^{-1}(E \cap F) = f^{-1}(E) \cap f^{-1}(F)$

Proof. For $x \in A$, we have

$$\begin{aligned} x \in f^{-1}(E \cap F) &\iff f(x) \in E \cap F \\ &\iff \Big[f(x) \in E\Big] \wedge \Big[f(x) \in F\Big] \\ &\iff \Big[x \in f^{-1}(E)\Big] \wedge \Big[x \in f^{-1}(F)\Big] \\ &\iff x \in f^{-1}(E) \cap f^{-1}(F). \end{aligned}$$

This completes the proof. □

Exercise 10.20 A function $f : \mathbb{Z} \to \mathbb{Z}$ is defined by $f(n) = 2n + 1$. Determine whether f is (a) injective, (b) surjective.

(a) We claim that f is injective. Suppose $f(n) = f(m)$ for $n, m \in \mathbb{Z}$. This means that $2n + 1 = 2m + 1$, or $2n = 2m$, or $n = m$. Since $f(n) = f(m)$ implies $n = m$, the function f is injective.

(b) We claim that f is not surjective. Consider $0 \in \mathbb{Z}$ in the codomain. For the sake of contradiction, suppose that there exist $n \in \mathbb{Z}$ such that $f(n) = 0$. This means $2n+1 = 0$ or $n = -\frac{1}{2}$ which is not an integer. This is the contradiction and proves the claim.

Exercise 10.24 Determine whether the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 4x + 9$ is (a) one-to-one, (b) onto.

(a) We claim that f is not one-to-one. Consider $x_1 = -3$ and $x_2 = -1$. Then $x_1 \neq x_2$ but

$$f(x_1) = (-3)^2 + 4(-3) + 9 = 6 = (-1)^2 + 4(-1) + 9 = f(-1).$$

(b) We claim that f is not surjective. Consider $y = 4$ in the codomain $\mathbb{R}$. Since

$$f(x) = (x+2)^2 + 5 \geq 5 \quad \text{for all} \quad x \in \mathbb{R},$$

there is no $x \in \mathbb{R}$ such that $f(x) = 4$. So 4 is not in the range, hence f is not surjective.

Exercise 10.32 Prove that the function $f : \mathbb{R} - \{2\} \to \mathbb{R} - \{5\}$ defined by $f(x) = \frac{5x+1}{x-2}$ is bijective.

Proof. We prove injectivity and surjectivity.

(i) Injectivity: Suppose that $f(x_1) = f(x_2)$ for $x_1, x_2 \in \mathbb{R} - \{2\}$. This means

$$\frac{5x_1+1}{x_1-2} = \frac{5x_2+1}{x_2-2}.$$

By multiplying $(x_1 - 2)(x_2 - 2)$ for both sides, we obtain

$$(5x_1+1)(x_2-2) = (5x_2+1)(x_1-2),$$

or

$$5x_1x_2 - 10x_1 + x_2 - 2 = 5x_1x_2 + x_1 - 10x_2 - 2,$$

or

$$-11x_1 = -11x_2,$$

or

$$x_1 = x_2.$$

This proves injectivity.

(ii) Surjectivity: Pick any element $y \in \mathbb{R} - \{5\}$ in the codomain. We want to find $x \in \mathbb{R} - \{2\}$ such that $f(x) = y$. We claim that

$$x = \frac{2y+1}{y-5}$$

works. Since $y \neq 5$, the expression makes sense and so $x \in \mathbb{R}$. Furthermore, $x \in \mathbb{R} - \{2\}$ because otherwise it implies

$$x = \frac{2y+1}{y-5} = 2 \implies 2y + 1 = 2(y-5) \implies 1 = -10$$

which is a contradiction. Finally, we check $f(x) = y$ by computation:

$$f(x) = \frac{5x+1}{x-2} = \frac{5\left(\frac{2y+1}{y-5}\right)+1}{\left(\frac{2y+1}{y-5}\right)-2} = \frac{5(2y+1)+(y-5)}{(2y+1)-2(y-5)} = \frac{11y}{11} = y.$$

This proves surjectivity.

□

Exercise 10.48 The composition $g \circ f : (0,1) \to \mathbb{R}$ of two functions f and g is given by $(g \circ f)(x) = \frac{4x-1}{2\sqrt{x-x^2}}$, where $f : (0,1) \to (-1,1)$ is defined by $f(x) = 2x - 1$ for $x \in (0,1)$. Determine the function g.

Solution. We use "a" right inverse function $f^{-1} : (-1,1) \to (0,1)$ of f. Define f^{-1} by

$$f^{-1}(x) = \frac{x+1}{2}.$$

This is well defined because for $-1 < x < 1$ we have $0 < \frac{x+1}{2} < 1$. It is easy to check that f^{-1} is in fact a right inverse by the calculation:

$$\big(f \circ f^{-1}\big)(x) = f\big(f^{-1}(x)\big) = f\big(\frac{x+1}{2}\big) = 2\left(\frac{x+1}{2}\right) - 1 = x.$$

In fact f is both sided inverse of f, hence "the" inverse of f, but we won't need this in what follows. To determine g by using $(g \circ f)$ and f^{-1} we use the associativity of the composition. Note that

$$(g \circ f) \circ f^{-1} = g \circ (f \circ f^{-1}) = g \circ \mathrm{id} = g.$$

Therefore, we have

$$\begin{aligned}
g(x) &= \Big((g \circ f) \circ f^{-1}\Big)(x) \\
&= (g \circ f)\big(f^{-1}(x)\big) \\
&= (g \circ f)\big(\frac{x+1}{2}\big) \\
&= \frac{4\left(\frac{x+1}{2}\right) - 1}{2\sqrt{\left(\frac{x+1}{2}\right) - \left(\frac{x+1}{2}\right)^2}} \\
&= \frac{2x+1}{\sqrt{(2x+2) - (x+1)^2}} \\
&= \frac{2x+1}{\sqrt{1-x^2}}.
\end{aligned}$$

This determines the function g. □