Partial Solutions for HW7

Exercise 10.50 Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 4x - 3$ is bijective and determine $f^{-1}(x)$ for $x \in \mathbb{R}$.

Solution. We first show injectivity. Suppose $f(x_1) = f(x_2)$ for $x_1, x_2 \in \mathbb{R}$. Then

$$4x_1 - 3 = 4x_2 - 3,$$

or

$$4x_1 = 4x_2,$$

or

$$x_1 = x_2.$$

Therefore f is injective. To show surjectivity, pick any element $y \in \mathbb{R}$ in the codomain. We want to find $x \in \mathbb{R}$ such that $f(x) = y$, or

$$4x - 3 = y.$$

By solving the linear equation for x, we have

$$x = \frac{y+3}{4} \in \mathbb{R}.$$

Therefore f is surjective. Also, above formula says

$$f^{-1}(y) = x = \frac{y+3}{4}.$$

Simply changing the dummy variable for a function, we obtain

$$f^{-1}(x) = \frac{x+3}{4}.$$

□

Exercise 10.56 Let A, B and C be nonempty sets and let f, g and h be functions such that $f : A \to B$ and $g, h : B \to C$. For each of the following, prove or disprove:

(a) If $g \circ f = h \circ f$, then $g = h$.

(b) If f is one-to-one and $g \circ f = h \circ f$, then $g = h$.

Solution. We claim that both (a) and (b) are false. Note that if one can find a counter example for (b) then it is automatically a counter example for (a).

Define A, B and C as

$$A = \{a\}, \quad B = \{b_1, b_2\}, \quad C = \{c_1, c_2\}.$$

Define a function f by $f(a) = c_1$. Define functions g, h by

$$g(b_1) = h(b_1) = c_1, \quad g(b_2) = c_1, \; h(b_2) = c_2.$$

Note that f is clearly one-to-one because there is only one element in the domain of f. We also have $g \circ f = h \circ f$ because they both map $a \in A$ to c_1. However, g and h are different because $g(b_2) \neq h(b_2)$. This counter example disproves (b) and also (a). □

Exercise 13.28 Prove that if 41 numbers are selected from the set $S = \{1, 2, ..., 50\}$, then there must exist two among those chosen whose sum is 80.

Proof. Denote a set of selected 41 numbers by X, i.e., X is a subset of S with cardinality 41. Since the biggest element of S is 50, no element smaller than 30 can be used to sum up to 80. Also, we exclude the case of $40 + 40 = 80$. Therefore possible elements in S to form a pair summing up to 80 are

$$\widetilde{S} := \{30, 31, \ldots, 39, 41, \ldots, 49, 50\} \subset S.$$

To increase the possibility of having a pair in X summing up to 80, we wish to have as much elements of $\widetilde{S}$ in X. We claim that at worst we have 11 elements in common, i.e.,

$$|\widetilde{S} \cap X| \geq 11.$$

This can be seen as follows:

$$|S| \geq |\widetilde{S} \cup X| = |\widetilde{S}| + |X| - |\widetilde{S} \cap X|$$

or

$$|\widetilde{S} \cap X| \geq |\widetilde{S}| + |X| - |S| = 20 + 41 - 50 = 11.$$

Now we consider a set

$$T := \Big\{\{30, 50\}, \{31, 49\}, \{32, 48\}, \ldots, \{39, 41\}\Big\}$$

which consists of sets of pairs summing up to 80. Let

$$f : \widetilde{S} \cap X \to T$$

be an obvious map defined by

$$f(n) = \{n, 80 - n\}.$$

Since $|\widetilde{S} \cap X| \geq 11$ and $|T| = 10$, by pigeon hole principle, we have

$$\exists\, n_1 \neq n_2 \in \widetilde{S} \cap X \quad s.t. \quad f(n_1) = f(n_2).$$

By definition of f, this means

$$\{n_1, 80 - n_1\} = \{n_2, 80 - n_2\}.$$

Since $n_1 \neq n_2$, this implies

$$n_1 = 80 - n_2,$$

equivalently

$$n_1 + n_2 = 80.$$

Note that $n_1, n_2 \in X$, in particular. Therefore we have found a pair of distinct elements in X summing up to 80. This completes the proof.

□

Exercise 13.34 Let F be a collection of subsets of $S = \{1, 2, ..., n\}$, $n \geq 2$, such that if $X, Y \in F$, then $X \cap Y = \emptyset$. Prove that $|F| \leq 2^{n-1}$.

Proof. To prove by contradiction, suppose $|F| > 2^{n-1}$. Consider a set

$$T := \Big\{ \{X, S - X\} : \ X \subseteq S \Big\}$$

consisting a set of pairs of two subsets of S complimentary to each other in S. Note that the cardinality of T is exactly half of the cardinality of the power set $\mathcal{P}(S)$ of S. Therefore

$$|T| = \frac{|\mathcal{P}(S)|}{2} = \frac{2^n}{2} = 2^{n-1}.$$

Consider a map

$$f : F \to T$$

defined by

$$f(X) := \{X, S - X\}.$$

Since $|F| > 2^{n-1} = |T|$, by pigeon hole principle, we have

$$\exists\, X \neq Y \in F \quad s.t. \quad f(X) = f(Y).$$

This means

$$\{X, S - X\} = \{Y, S - Y\}.$$

Since $X \neq Y$, this forces that $X = S - Y$. In particular, we have

$$X, Y \in F \quad s.t. \quad X \cap Y = \emptyset.$$

This contradicts the assumption of the problem. This completes the proof. □

Exercise 11.8 Prove that the function $f : \mathbb{N} \to \mathbb{Z}$ defined by $f(n) = \frac{1+(-1)^n(2n-1)}{4}$ is bijective.

Proof. We start by analyzing the function f by dividing $\mathbb{N}$ into even and odd parts. For odd number $n = 2k + 1$ with $k \geq 0$, we have

$$f(2k+1) = \frac{1 + (-1)^{2k+1}\big(2(2k+1) - 1\big)}{4} = \frac{1 - (4k+1)}{4} = -k.$$

For even number $n = 2k$ with $k \geq 1$, we have

$$f(2k) = \frac{1 + (-1)^{2k}\big(2(2k) - 1\big)}{4} = \frac{1 + (4k-1)}{4} = k.$$

Now we show that f is injective. Suppose that $f(n_1) = f(n_2)$ for some $n_1, n_2 \in \mathbb{N}$. We divide the case into two depending on the sign of the value $f(n_1) = f(n_2)$. Assume that $f(n_1) = f(n_2) \leq 0$. Then this forces n_1, n_2 to be odd number by the above observation, hence

$$n_1 = 2k_1 + 1, \quad n_2 = 2k_2 + 1.$$

But then

$$-k_1 = f(n_1) = f(n_2) = -k_2,$$

and so

$$k_1 = k_2,$$

and so

$$n_1 = n_2.$$

Similarly, if $f(n_1) = f(n_2) \geq 1$, then this forces n_1, n_2 to be even number by the above observation, hence

$$n_1 = 2k_1, \quad n_2 = 2k_2.$$

But then

$$k_1 = f(n_1) = f(n_2) = k_2,$$

and so

$$n_1 = n_2.$$

Since we obtained $n_1 = n_2$ in either case, this proves the injectivity of f.

Now we prove surjectivity of f. We need to show that every integers are in the image of f. If $k \leq 0$, then

$$f(2(-k)+1) = k$$

where $2(-k)+1 \in \mathbb{N}$. Similarly, if $k \geq 1$, then

$$f(2k) = k$$

where $2k \in \mathbb{N}$. Therefore f is surjective.

□