Partial Solutions for HW8

Exercise 11.10 Let A be a denumerable set and let $B = \{x, y\}$. Prove that $A \times B$ is denumerable.

Proof.
Method 1: Since A is denumerable, there is a bijection $f : \mathbb{N} \to A$. In other words, we can list the elements of the set A as follows:

$$A = \{a_1, a_2, a_3 \cdots\}.$$

Using this description of a set A, we can list the elements of the set $A \times B$ as

$$A \times B = \{(a_1, x), (a_1, y), (a_2, x), (a_2, y), (a_3, x), (a_3, y), \cdots\}.$$

This proves that $A \times B$ is denumerable.

Method 2: We can write $A \times B$ as a disjoint union of the following two sets

$$A \times \{x\}, \quad A \times \{y\}.$$

It is clear that $A \times \{x\}$ is numerically equivalent to a set A through the bijection g defined by

$$g : A \to A \times \{x\}, \quad a \mapsto (a, x).$$

Similarly, $A \times \{y\}$ is also numerically equivalent to a set A. Therefore $A \times B$ is a union of two countable sets, hence countable. (In class, we have proven that even countable union of countable sets is countable.) Since $A \times B$ is infinite countable set, it is denumerable. □

Exercise 11.12 Prove that the set of all 2-element subsets of $\mathbb{N}$ is denumerable.

Proof. Any 2-element subsets of $\mathbb{N}$ can be uniquely written as

$$\{a, b\} \text{ for some } a, b \in \mathbb{N} \;\; s.t. \;\; a < b.$$

Therefore the set of all 2-element subsets of $\mathbb{N}$ is in bijection to the set

$$S := \{(a, b) \in \mathbb{N} \times \mathbb{N} \, : \, a < b\}.$$

Note that S is naturally a subset of $\mathbb{N} \times \mathbb{N}$ which is proven to be countably infinite in class. Since S is an infinite subset of denumerable set $\mathbb{N} \times \mathbb{N}$, S is also denumerable. This completes the proof. □

Exercise 11.18 A function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is defined by $f(m, n) = 2^{m-1}(2n - 1)$.

(a) Prove that f is one-to-one and onto.

(b) Show that $\mathbb{N} \times \mathbb{N}$ is denumerable.

Proof. We start by proving injectivity of f. Suppose that

$$f(m_1, n_1) = f(m_2, n_2) \quad \text{for some} \quad (m_1, n_1), (m_2, n_2) \in \mathbb{N} \times \mathbb{N}.$$

By definition of f, we have

$$2^{m_1-1}(2n_1 - 1) = 2^{m_2-1}(2n_2 - 1).$$

Without loss of generality, we may assume $m_1 \leq m_2$. Dividing both sides by 2^{m_1-1}, we obtain

$$(2n_1 - 1) = 2^{m_2-m_1}(2n_2 - 1) \in \mathbb{N}.$$

Since LHS is odd number, RHS can not have any factor of 2. Therefore $m_2 - m_1 = 0$, or

$$m_1 = m_2.$$

Plugging this equality to the previous equation, we obtain

$$2n_1 - 1 = 2n_2 - 1,$$

or

$$n_1 = n_2.$$

Therefore we have $(m_1, n_1) = (m_2, n_2)$, hence f is injective.

Now we prove surjectivity of f. Let k be any element in the codomain $\mathbb{N}$. Define m be the largest natural number such that 2^{m-1} divides k. There exist such m because k cannot be divided by infinitely many 2 factors. By definition of m, we have

$$k = 2^{m-1} \cdot \ell \quad \text{for some} \quad \ell \in \mathbb{N}.$$

If ℓ were even number, we can pull out at least one more 2 factor from ℓ which contradicts the choice of m. Therefore ℓ is an odd number hence

$$\ell = 2n - 1 \quad \text{for some} \quad n \in \mathbb{N}.$$

Thus

$$k = 2^{m-1}(2n - 1),$$

and this proves surjectivity of f. $\square$

Exercise 11.20 Prove that the set of irrational numbers is uncountable.

Proof. We prove by contradiction. Suppose that the set $\mathbb{R} - \mathbb{Q}$ of irrational numbers is countable. We know that the set $\mathbb{Q}$ of rational numbers is countable. Therefore the set of real numbers

$$\mathbb{R} = (\mathbb{R} - \mathbb{Q}) \cup \mathbb{Q}$$

is a union of two countable sets, hence countable. This contradicts the fact that $\mathbb{R}$ is uncountable. Thus, the set of irrational numbers is uncountable. $\square$

Exercise 11.30 Let $A = \{a, b, c\}$. Then $\mathcal{P}(A)$ consists of the following subsets of A:

$$A_a = \emptyset, \quad A_b = A, \quad A_c = \{a, b\}, \quad A_d = \{a, c\},$$
$$A_e = \{b, c\}, \quad A_f = \{a\}, \quad A_g = \{b\}, \quad A_h = \{c\}.$$

In one part of the proof of Theorem 11.17, it was established (using a contradiction argument) that $|A| < |\mathcal{P}(A)|$ for every nonempty set A. In this argument, the existence of a bijective function $g : A \to \mathcal{P}(A)$ is assumed, where $g(x) = A_x$ for each $x \in A$. Then a subset B of A is defined by

$$B = \{x \in A \,:\, x \notin A_x\}$$

(a) For the sets A and $\mathcal{P}(A)$ described above, what is the set B?

(b) What does the set B in (a) illustrate?

Solution. By definition, B is the set of all elements $x \in A$ satisfying the condition

$$x \notin A_x.$$

We check this condition for each a, b, and c.

1. $a \notin A_a = \emptyset$.
2. $b \in A_b = A$.
3. $c \notin A_c = \{a, b\}$.

Since a, c satisfy the condition but b doesn't, we have

$$B = \{a, c\}.$$

We can read from the list of all subsets of A above that

$$B = A_d.$$

This example illustrates by a concrete example that there is no surjective function $g : A \to \mathcal{P}(A)$. To be more explicit, we were given a function

$$g : A \to \mathcal{P}(A)$$

defined by

$$g(a) = A_a, \quad g(b) = A_b, \quad g(c) = A_c.$$

A function g is not surjective because we found an element $B = A_d \in \mathcal{P}(A)$ which is not in the image of g. $\square$