Partial Solutions for HW9

**Exercise 6.8** Find a formula for \(1 + 4 + 7 + \cdots + (3n - 2)\) for positive integers \(n\) and then verify your formula by mathematical induction.

*Solution.* We first make a guess and then prove it by mathematical induction. Note that

\[
1 + 4 + 7 + \cdots + (3n - 2) = \sum_{i=1}^{n} (3i - 2) = 3 \sum_{i=1}^{n} i - 2 \sum_{i=1}^{n} 1 = 3 \cdot \frac{n(n+1)}{2} - 2n = \frac{n(3n-1)}{2}.
\]

In fact, the above lines are more or less the proof. But we will still verify it with mathematical induction now. Since

\[
1 = 1 \cdot \frac{(3 \cdot 1 - 1)}{2},
\]

the statement is true for \(n = 1\). Assume that

\[
1 + 4 + 7 + \cdots + (3k - 2) = \frac{k(3k - 1)}{2},
\]

where \(k\) is a positive integer. Then we have

\[
1 + 4 + 7 + \cdots + (3k - 2) + (3(k+1) - 2) = \frac{k(3k - 1)}{2} + (3k + 1) = \frac{3k^2 + 5k + 2}{2} = \frac{(k+1)(3k+2)}{2} = \frac{(k+1)(3(k+1) + 2)}{2},
\]

which proves the statement for \(k + 1\). \(\square\)

**Exercise 6.16** Prove that \(7 \mid 3^{4n+1} - 5^{2n-1}\) for every positive integer \(n\).

*Proof.* We prove by mathematical induction. Since

\[
3^5 - 5^1 = 243 - 5 = 238 = 7 \cdot 34,
\]

the statement is true for \(n = 1\). Assume that

\[
7 \mid 3^{4k+1} - 5^{2k-1},
\]
where $k$ is a positive integer. That is,
\[ \exists m \in \mathbb{Z} \text{ s.t. } 3^{4k+1} - 5^{2k-1} = 7m. \]

Therefore, we have
\[
3^{4(k+1)+1} - 5^{2(k+1)-1} = 3^4 \cdot 3^{4k+1} - 5^2 \cdot 5^{2k-1} \\
= 81(5^{2k-1} + 7m) - 25 \cdot 5^{2k-1} \\
= (81 - 25)5^{2k-1} + 7 \cdot 81 \cdot m \\
= 7\left(8 \cdot 5^{2k-1} + 81m\right).
\]

This means that
\[ 7 \mid 3^{4(k+1)+1} - 5^{2(k+1)-1}, \]
which completes the proof. \(\square\)

**Exercise 6.24** Prove Bernoulli’s Identity: For every real number $x > -1$ and every positive integer $n$,
\[(1 + x)^n \geq 1 + nx.\]

*Proof.* We prove by mathematical induction. Since
\[(1 + x) \geq 1 + x \quad \forall x \in \mathbb{R}\]
the statement is true for $n = 1$. Assume that
\[(1 + x)^k \geq 1 + kx \quad \forall x > -1\]
where $k$ is a positive integer. For any $x > -1$, we have
\[
(1 + x)^{k+1} = (1 + x)(1 + x)^k \\
\geq (1 + x)(1 + kx) \\
= 1 + (1 + k)x + kx^2 \\
\geq 1 + (1 + k)x.
\]

We have used the fact that $x > -1$ or $(1 + x) > 0$ for the second line. This proves the statement for $(k + 1)$, which completes the proof. \(\square\)

**Exercise 6.34** A sequence $\{a_n\}$ is defined recursively by $a_1 = 1$, $a_2 = 2$ and $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 3$. Conjecture a formula for $a_n$ and verify that your conjecture is correct.

*Solution.* To make a guess, we compute the first few terms:
\[
\begin{align*}
a_1 &= 1 \\
a_2 &= 2 \\
a_3 &= 2 + 2 \cdot 1 = 4 \\
a_4 &= 4 + 2 \cdot 2 = 8 \\
a_5 &= 8 + 2 \cdot 4 = 16.
\end{align*}
\]
It is then natural to conjecture 
\[ \forall n \in \mathbb{N} \quad a_n = 2^{n-1}. \]
We prove this conjecture by mathematical induction. Define statement 
\[ P(n) : \ a_n = 2^{n-1}. \]
We will prove the followings:

(i) \( P(1), P(2) \) are true.

(ii) \( P(1) \land P(2) \land \cdots \land P(n) \implies P(n + 1) \) for all natural numbers \( n \geq 2 \).

By computation above, \( P(1) \) and \( P(2) \) are true. Now we assume \( P(1), P(2), \cdots, P(n) \) where \( n \) is a positive integer greater than equal to 2. Then we have

\[
a_{n+1} = a_n + 2a_{n-1} \\
= 2^{n-1} + 2 \cdot 2^{n-2} \\
= 2^{n-1} + 2^{n-1} \\
= 2^n,
\]

where we used the statements \( P(n - 1) \) and \( P(n) \) for the second line. This proves \( P(n + 1) \) and completes the proof. \( \square \)

**Exercise 6.36** Consider the sequence \( F_1, F_2, F_3, \ldots \) where

\[
F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5 \text{ and } F_6 = 8.
\]

The terms of this sequence are called Fibonacci numbers.

(a) Define the sequence of Fibonacci numbers by means of a recurrence relation.

(b) Prove that \( 2 \mid F_n \) if and only if \( 3 \mid n \).

**Proof.** We define the sequence of Fibonacci numbers as

1. Initial condition: \( F_1 = F_2 = 1 \).

2. Recurrence relation: \( F_n = F_{n-1} + F_{n-2} \) for all \( n \geq 3 \).

Define statement 
\[ P(n) : \ 2 \mid F_n \iff 3 \mid n. \]
We will prove the followings:

(i) \( P(1), P(2), P(3) \) are true.

(ii) \( P(1) \land P(2) \land \cdots \land P(n) \implies P(n + 1) \) for all natural numbers \( n \geq 3 \).
Note that $P(1)$, $P(2)$ and $P(3)$ are true because $F_3 = 2$ is the only even number out of $F_1$, $F_2$, $F_3$. Now we assume $P(1), P(2), \cdots, P(n)$ where $n$ is a positive integer greater than equal to 3. We have

$$F_{n+1} = F_n + F_{n-1}$$
$$= \left(F_{n-1} + F_{n-2}\right) + F_{n-1}$$
$$= F_{n-2} + 2 \cdot F_{n-1},$$

which implies that the parity of $F_{n+1}$ and $F_{n-2}$ is same. Therefore, we conclude

$$2 \mid F_{n+1} \iff 2 \mid F_{n-2}$$
$$\iff 3 \mid (n - 2)$$
$$\iff 3 \mid (n - 2) + 3$$
$$\iff 3 \mid (n + 1).$$

This proves $P(n + 1)$ and completes the proof. □