Partial Solutions for HW9

Exercise 6.8 Find a formula for $1+4+7+\cdots+(3n-2)$ for positive integers n and then verify your formula by mathematical induction.

Solution. We first make a guess and then prove it by mathematical induction. Note that

$$\begin{aligned} 1+4+7+\cdots+(3n-2) &= \sum_{i=1}^{n}(3i-2) \\ &= 3\sum_{i=1}^{n} i - 2\sum_{i=1}^{n} 1 \\ &= 3\cdot\frac{n(n+1)}{2} - 2n \\ &= \frac{n(3n-1)}{2}. \end{aligned}$$

In fact, the above lines are more or less the proof. But we will still verify it with mathematical induction now. Since

$$1 = \frac{1\cdot(3\cdot 1-1)}{2},$$

the statement is true for $n=1$. Assume that

$$1+4+7+\cdots+(3k-2) = \frac{k(3k-1)}{2},$$

where k is a positive integer. Then we have

$$\begin{aligned} 1+4+7+\cdots+(3k-2)+\big(3(k+1)-2\big) &= \frac{k(3k-1)}{2} + (3k+1) \\ &= \frac{3k^2+5k+2}{2} \\ &= \frac{(k+1)(3k+2)}{2} \\ &= \frac{(k+1)\big(3(k+1)+2\big)}{2}, \end{aligned}$$

which proves the statement for $k+1$. □

Exercise 6.16 Prove that $7 \mid 3^{4n+1} - 5^{2n-1}$ for every positive integer n.

Proof. We prove by mathematical induction. Since

$$3^5 - 5^1 = 243 - 5 = 238 = 7\cdot 34,$$

the statement is true for $n=1$. Assume that

$$7 \mid 3^{4k+1} - 5^{2k-1},$$

where k is a positive integer. That is,

$$\exists\, m \in \mathbb{Z} \quad s.t. \quad 3^{4k+1} - 5^{2k-1} = 7m.$$

Therefore, we have

$$\begin{aligned} 3^{4(k+1)+1} - 5^{2(k+1)-1} &= 3^4 \cdot 3^{4k+1} - 5^2 \cdot 5^{2k-1} \\ &= 81(5^{2k-1} + 7m) - 25 \cdot 5^{2k-1} \\ &= (81-25)5^{2k-1} + 7 \cdot 81 \cdot m \\ &= 7\Big(8 \cdot 5^{2k-1} + 81m\Big). \end{aligned}$$

This means that

$$7 \,|\, 3^{4(k+1)+1} - 5^{2(k+1)-1},$$

which completes the proof. □

Exercise 6.24 Prove Bernoulli's Identity: For every real number $x > -1$ and every positive integer n,

$$(1+x)^n \geq 1 + nx.$$

Proof. We prove by mathematical induction. Since

$$(1+x) \geq 1 + x \quad \forall x \in \mathbb{R}$$

the statement is true for $n = 1$. Assume that

$$(1+x)^k \geq 1 + kx \quad \forall x > -1$$

where k is a positive integer. For any $x > -1$, we have

$$\begin{aligned} (1+x)^{k+1} &= (1+x)(1+x)^k \\ &\geq (1+x)(1+kx) \\ &= 1 + (1+k)x + kx^2 \\ &\geq 1 + (1+k)x. \end{aligned}$$

We have used the fact that $x > -1$ or $(1+x) > 0$ for the second line. This proves the statement for $(k+1)$, which completes the proof. □

Exercise 6.34 A sequence $\{a_n\}$ is defined recursively by $a_1 = 1, a_2 = 2$ and $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 3$. Conjecture a formula for an and verify that your conjecture is correct.

Solution. To make a guess, we compute the first few terms:

$$\begin{aligned} a_1 &= 1 \\ a_2 &= 2 \\ a_3 &= 2 + 2 \cdot 1 = 4 \\ a_4 &= 4 + 2 \cdot 2 = 8 \\ a_5 &= 8 + 2 \cdot 4 = 16. \end{aligned}$$

It is then natural to conjecture

$$\forall n \in \mathbb{N} \quad a_n = 2^{n-1}.$$

We prove this conjecture by mathematical induction. Define statement

$$P(n): \; a_n = 2^{n-1}.$$

We will prove the followings:

(i) $P(1), P(2)$ are true.

(ii) $P(1) \wedge P(2) \wedge \cdots \wedge P(n) \implies P(n+1)$ for all natural numbers $n \geq 2$.

By computation above, $P(1)$ and $P(2)$ are true. Now we assume $P(1), P(2), \cdots, P(n)$ where n is a positive integer greater than equal to 2. Then we have

$$\begin{aligned} a_{n+1} &= a_n + 2a_{n-1} \\ &= 2^{n-1} + 2 \cdot 2^{n-2} \\ &= 2^{n-1} + 2^{n-1} \\ &= 2^n, \end{aligned}$$

where we used the statements $P(n-1)$ and $P(n)$ for the second line. This proves $P(n+1)$ and completes the proof. □

Exercise 6.36 Consider the sequence $F_1, F_2, F_3, \ldots$ where

$$F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5 \text{ and } F_6 = 8.$$

The terms of this sequence are called Fibonacci numbers.

(a) Define the sequence of Fibonacci numbers by means of a recurrence relation.

(b) Prove that $2 \mid F_n$ if and only if $3 \mid n$.

Proof. We define the sequence of Fibonacci numbers as

1. Initial condition: $F_1 = F_2 = 1$.
2. Recurrence relation: $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 3$.

Define statement

$$P(n) \; : \; 2 \mid F_n \iff 3 \mid n.$$

We will prove the followings:

(i) $P(1), P(2), P(3)$ are true.

(ii) $P(1) \wedge P(2) \wedge \cdots \wedge P(n) \implies P(n+1)$ for all natural numbers $n \geq 3$.

Note that $P(1)$, $P(2)$ and $P(3)$ are true because $F_3 = 2$ is the only even number out of F_1, F_2, F_3. Now we assume $P(1), P(2), \cdots, P(n)$ where n is a positive integer greater than equal to 3. We have

$$\begin{aligned} F_{n+1} &= F_n + F_{n-1} \\ &= \Big(F_{n-1} + F_{n-2}\Big) + F_{n-1} \\ &= F_{n-2} + 2 \cdot F_{n-1}, \end{aligned}$$

which implies that the parity of F_{n+1} and F_{n-2} is same. Therefore, we conclude

$$\begin{aligned} 2 \mid F_{n+1} &\iff 2 \mid F_{n-2} \\ &\iff 3 \mid (n-2) \\ &\iff 3 \mid (n-2) + 3 \\ &\iff 3 \mid (n+1). \end{aligned}$$

This proves $P(n+1)$ and completes the proof. □