LECTURE 13

(Fri. FEB. 7, 2020)

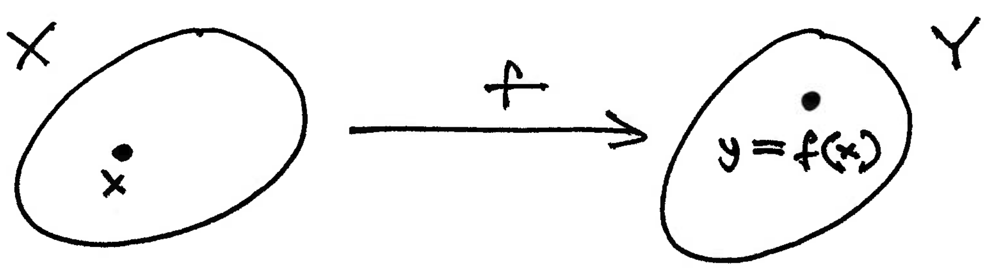
Functions (Chapter 10).

1) Vague definition: Given two sets,

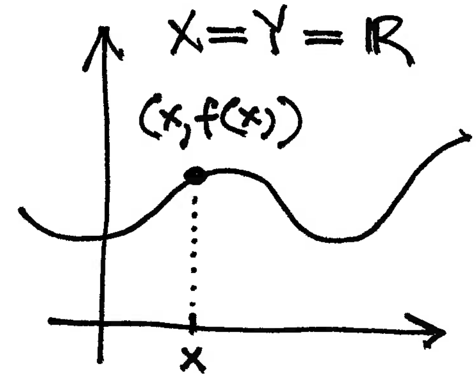
$X =$ allowable inputs ("domain")

$Y =$ allowable outputs ("codomain")

f is a machine/rule which transforms an element $x \in X$ into an element $f(x) \in Y$.



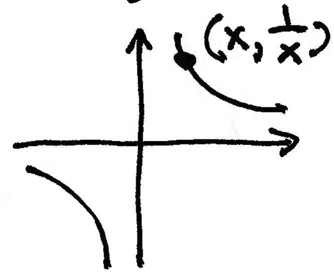
The graph of $f: X \to Y$ is the set of pairs (x, y) where $y = f(x)$.



Ex $X = \mathbb{R}^{\times} = \mathbb{R} - \{0\}$
(nonzero real x)

$Y = \mathbb{R}$

$f(x) = \frac{1}{x}$



(could've taken $Y = \mathbb{R}^{\times}$ but not $Y = (0, \infty)$.)

$f = (X, Y, R)$.

2) ~~Formal~~ (actual) definition:

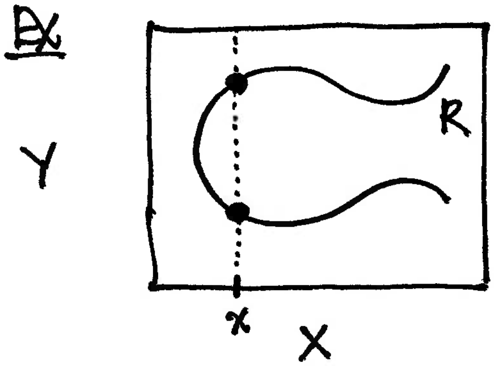
- A "function" from X to Y is a relation $R \subseteq X \times Y$ (subset) with the property:

  $\forall x \in X$ there's a unique $y \in Y$ such that $(x, y) \in R$.

Notation: $f(x)$ denotes that y. — say f maps x to y: $f : x \longmapsto y$.

- The subset R is called the graph of f.

Ex



not a function:

- each "vertical line" $\{(x,y) : y \in Y\}$ must intersect R in exactly one point.

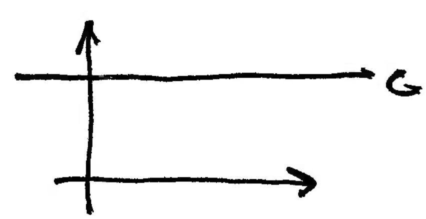
Remark: We're not saying every $y \in Y$ must be of the form $f(x)$ for some $x \in X$.

- If this happens we say f is "surjective". I.e.,

$$\forall y \in Y \ \exists x \in X : \ y = f(x)$$

(meaning the equation $y = f(x)$ has a solution $x \in X$ for every choice of $y \in Y$)

think of "constant" function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = c$.

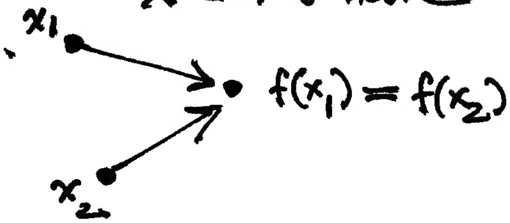


- "*injective*" means the eqn. $y = f(x)$ has at most one solution $x \in X$, for every choice of $y \in Y$.
  - I.e., for any two $x_1, x_2 \in X$,

$$f(x_1) = f(x_2) \implies x_1 = x_2$$

(in other words, two distinct $x_1 \neq x_2$ cannot collapse to the same point in Y).

~ can't have:



- "*bijective*" means surjective & injective.
  Equivalently, $\forall y \in Y$ there's a unique solution $x \in X$ to the eqn.

$$y = f(x). \quad (*)$$

— When f is bijective, it therefore makes sense to define its "inverse function"

$$f^{-1} : Y \longrightarrow X$$

$$f^{-1}(y) = \text{the solution } x \in X \text{ to the eqn. } (*)$$

[has graph

$$R^{-1} = \{(y,x) : (x,y) \in R\}.]$$

Rk.
Think of bijective f as a dictionary.
Each $x \in X$ corresponds to a unique $y \in Y$.
~ pair elements of X, Y
("one-to-one correspondence")
$\implies |X| = |Y|$ if finite.