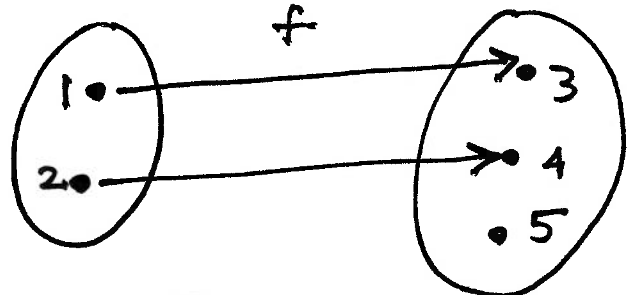
LECTURE 14
(Mon. FEB. 10, 2020)

Ex. $X = \{1, 2\}$ and $Y = \{3, 4, 5\}$.

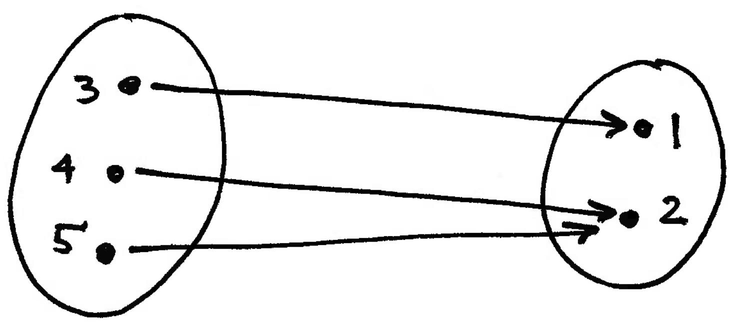
Def. $f: X \to Y$ by $f(1) = 3$, $f(2) = 4$.



no collapsing, but 5 isn't hit. So this f is <u>injective</u>, not surjective.

Def. $g: Y \to X$ by

$g(3) = 1$, $g(4) = 2$, $g(5) = 2$.



collapsing, maps onto <u>all</u> of X. So this g is <u>surjective</u>, not injective.

($g(4) = g(5)$, although $4 \neq 5$)

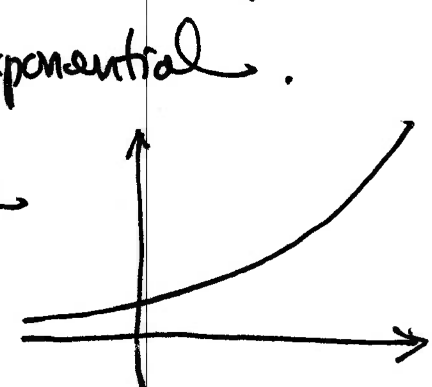
Ex. $X = Y = \mathbb{R}$. Let $f: \mathbb{R} \to \mathbb{R}$ exponential.

$f(x) = e^x$

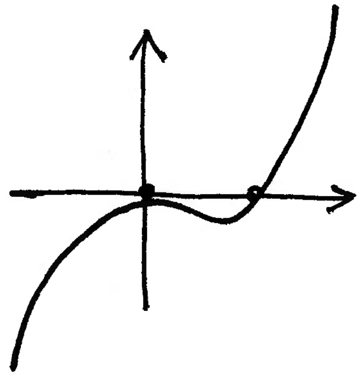
<u>injective</u>, not surjective

↑

all values are > 0.



$g(x) = x^3 - x^2 = x^2(x-1)$ surjective, not injective



$g(0) = g(1) = 0$.

[constant function
$h(x) = c$
neither surjective
nor injective]

Ex A $m \times n$-matrix, linear transformation $f: \mathbb{R}^n \to \mathbb{R}^m$
$f(x) = Ax$.

- f surjective $\Longleftrightarrow$ columns of A span $\mathbb{R}^m$.

- f injective $\Longleftrightarrow$ columns of A are linearly independent.

- f bijective $\Longleftrightarrow$ columns of A form a basis for $\mathbb{R}^m$
(i.e., $m = n$ and A is invertible)

- In this case $f^{-1}(y) = A^{-1}y$
A^{-1} = the inverse matrix.

<u>Composition</u> (10.4) Given two functions

$$f : X \longrightarrow Y \quad \text{and} \quad g : Y \longrightarrow Z$$

(domain$(g) = Y =$ codomain(f)).

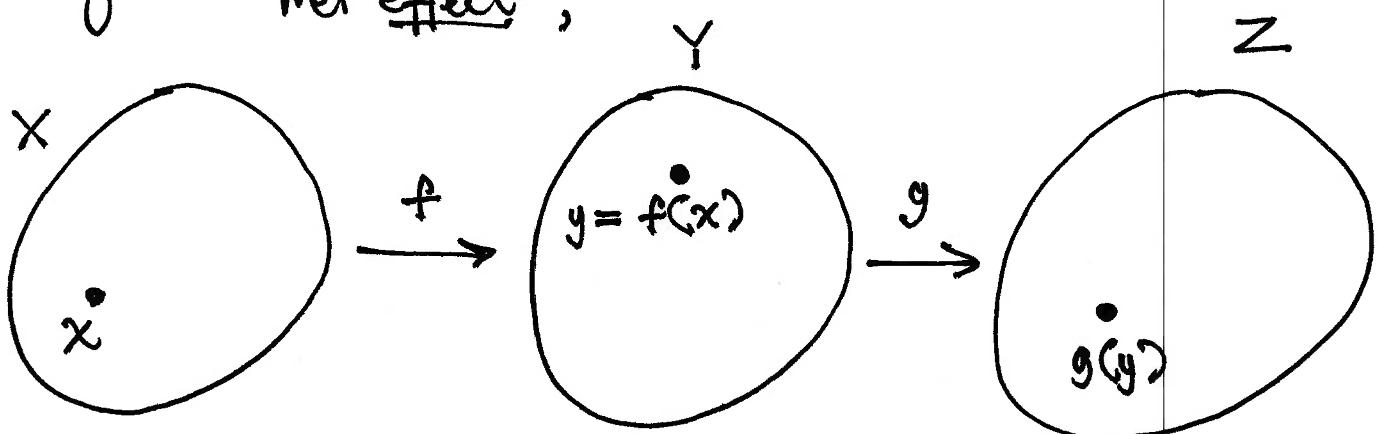
They're composable: Their <u>composition</u> is the function

$$g \circ f : X \longrightarrow Z$$

~ read $g \circ f$ from <u>right</u> to <u>left</u>...

$$(g \circ f)(x) = g\,(f(x))$$

– Diagram: "Net <u>effect</u>",



[potentially <u>confusing</u>, but <u>actual</u> def.:

– The <u>graph</u> of $g \circ f$ is the set of pairs $(x, z) \in X \times Z$ for which $\exists\, y \in Y$:

$$(x, y) \in R_f \ \underline{\text{and}}\ (y, z) \in R_g \ . \ ]$$

Ex. ($X = Y = Z = (0,\infty)$)

$f(x) = \frac{1}{x}$

$g(x) = e^x$

Here $(g \circ f)(x) = g(\frac{1}{x}) = \exp(\frac{1}{x})$

$(f \circ g)(x) = f(e^x) = \exp(-x)$.

[shows $f \circ g \neq g \circ f$ in general, (even when both sides are defined)].

$x = 1: \; e \neq \frac{1}{e}$

Check: If f, g, h are three composable functions,

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

(why? Both sides send an $x \in X$ to $h(g(f(x)))$.)

Thm. Supp. f, g are composable functions (as above). Then:

(1) f, g surjective $\Rightarrow$ $g \circ f$ surjective

(2) f, g injective $\Rightarrow$ $g \circ f$ injective

(3) f, g bijective $\Rightarrow$ $g \circ f$ bijective.

PROOF. (1) Given any $z \in Z$ we must find an $x \in X$ s.t. $(g \circ f)(x) = z$.

Since g is surjective, we can find $y \in Y$ s.t. $g(y) = z$.

Since f is <u>also</u> surjective, we can find an $x \in X$ s.t. $f(x) = y$. But then this x works: Indeed,

$$(g \circ f)(x) = g(f(x)) = g(y) = z. \checkmark$$

(2) Let $x_1, x_2 \in X$ map to the same element in Z:
(via $g \circ f$)

$$(g \circ f)(x_1) = (g \circ f)(x_2).$$

Must deduce $x_1 = x_2$. The above eqn. amounts to:

$$g(f(x_1)) = g(f(x_2)).$$

Since g <u>is</u> injective, this allows us to conclude the two inputs are the <u>same</u>:

$$f(x_1) = f(x_2).$$

Since f is <u>also</u> injective, this implies $x_1 = x_2$ as desired. ✓

(3) Immediate from (1) & (2):

"<u>Bijective</u>" means surjective & injective.

$\square$