LECTURE 15

(Wed. FEB. 12, 2020)

- Recall: A bijective $f : X \to Y$ has an "inverse function" $f^{-1} : Y \to X$

$$f^{-1}(y) = \text{the } x \in X \text{ mapping to } y \text{ via } f.$$

In other words, $f(f^{-1}(y)) = y$ and
$f^{-1}(f(x)) = x$.

reformulate:

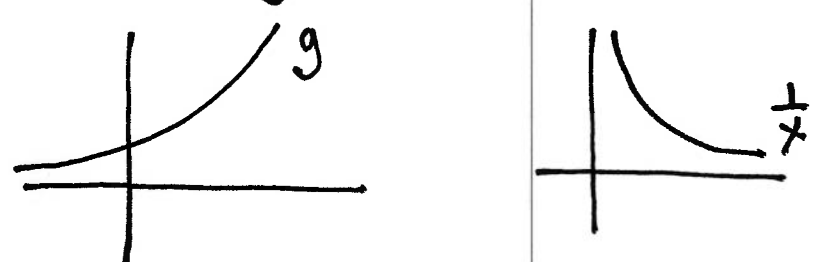
$$f \circ f^{-1} = Id_Y \quad \text{and} \quad f^{-1} \circ f = Id_X$$

[Here $Id_X : X \to X$ is the identity function
$Id_X(x) = x$.].

EX (cont) $f(x) = \frac{1}{x}$, $g(x) = e^x$

f is ~~both are~~ bijective $(0,\infty) \to (0,\infty)$

graphs:



g is bijective $\mathbb{R} \to (0,\infty)$.

- inverse?

$f^{-1} = f$

$g^{-1} = \ln$

(indeed $(f \circ f)(x) = f(\frac{1}{x}) = \frac{1}{\frac{1}{x}} = x$

and $(\exp \circ \ln)(x) = e^{\ln x} = x$

$(\ln \circ \exp)(x) = \ln(e^x) = x$)

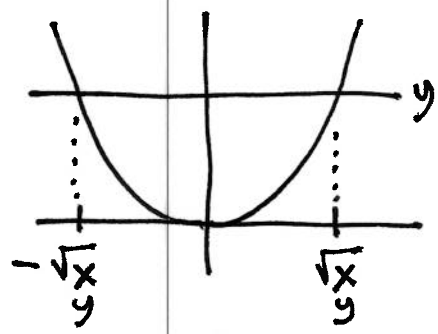
Stress: When f not bijective, f^{-1} doesn't make sense!

EX $f: \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x^2$

(not surj., nor inj.)

- Here $f(x) = y$ has two
(say $y > 0$) solutions $x = \pm\sqrt{y}$.

~ which one should be $f^{-1}(y)$?



If we declare that

$$f^{-1}(y) := \sqrt{y} \quad \text{(the positive solution)}$$

Then

$$f(f^{-1}(y)) = \sqrt{y}^2 = y$$

BUT

$$f^{-1}(f(x)) = \sqrt{x^2} = |x| \neq x \text{ for } x < 0.$$

- So, viewing f as a function $\mathbb{R} \longrightarrow [0, \infty)$ it
(surjective)
has a right inverse, but not a two-sided inverse.

- However, viewing f as a function $(0, \infty) \longrightarrow (0, \infty)$
(bijective)
it does have f^{-1}

("the pos. squareroot")

EXC: Function $f: X \longrightarrow Y$.

- f is surjective $\Longleftrightarrow$ f has a right inverse ($f \circ g = \mathrm{Id}_Y$).
- f is injective $\Longleftrightarrow$ f has a left inverse ($g \circ f = \mathrm{Id}_X$)

(This g is not unique in general, cf. above ex.: $\pm\sqrt{y}$)

Let $f: X \to Y$ be a function.

<u>Def</u> (1) For every $A \subseteq X$ its <u>image</u> (under f) is the subset $f(A) \subseteq Y$ def. by

$$f(A) = \{y \in Y : y = f(a) \text{ for some } a \in A\}.$$
$$= \{f(a) : a \in A\}.$$

(2) For every $C \subseteq Y$ its <u>inverse image</u> (via f) is the subset $f^{-1}(C) \subseteq X$ given by

$$f^{-1}(C) = \{x \in X : f(x) \in C\}.$$

<u>Note</u>: $f^{-1}(C)$ makes sense even if f <u>not</u> bijective (i.e., there's no f^{-1}).

- when f <u>is</u> bijective, $f^{-1}(C)$ equals the <u>image</u> of C under f^{-1} (so no <u>conflict</u> in notation)

<u>Image</u>: