LECTURE 15
(Wed. FEB. 12, 2020)
Recall: A bijective \( f: X \rightarrow Y \) has an "inverse function" \( f^{-1}: Y \rightarrow X \)

\[ f^{-1}(y) = \text{the } x \in X \text{ mapping to } y \text{ via } f. \]

In other words, \( f(f^{-1}(y)) = y \) and \( f^{-1}(f(x)) = x. \)

Reformulate:

\[ f \circ f^{-1} = \text{Id}_Y \text{ and } f^{-1} \circ f = \text{Id}_X. \]

[Here, \( \text{Id}_X: X \rightarrow X \) is the identity function \( \text{Id}_X(x) = x. \).]

**Example (cont):** \( f(x) = \frac{1}{x} \) \( g(x) = e^x \)

\[ f \text{ is both one-to-one bijective } (0, \infty) \rightarrow (0, \infty) \]

Graphs:

\( g \) is bijective \( \mathbb{R} \rightarrow (0, \infty) \)

\( f^{-1} = f \)

\( g^{-1} = \ln \)

(Indeed \( (f \circ f)(x) = f(\frac{1}{x}) = \frac{1}{x} = x \)

and \( (\exp \circ \ln)(x) = e^{\ln x} = x \)

\( (\ln \circ \exp)(x) = \ln(e^x) = x \)
Stress: When \( f \) not bijective, \( f^{-1} \) doesn't make sense!

**Ex:** \( f: \mathbb{R} \to \mathbb{R}, \ f(x) = x^2 \)  
- Here \( f(x) = y \) has two solutions \( x = \pm \sqrt{y} \).  
  - Which one should be \( f^{-1}(y) \)?

If we declare that  
\[ f^{-1}(y) := \sqrt{y} \ (\text{the positive solution}) \]

Then  
\[ f(f^{-1}(y)) = \sqrt{y^2} = y \]

But  
\[ f^{-1}(f(x)) = \sqrt{x^2} = |x| \neq x \text{ for } x < 0. \]

- So, viewing \( f \) as a function \( \mathbb{R} \to [0, \infty) \) it has a right inverse, but not a two-sided inverse.

- However, viewing \( f \) as a function \( (0, \infty) \to (0, \infty) \) it does have \( f^{-1} \) (bijective)  
  \(" the pos. square root" \)
Exercise: Function $f : X \rightarrow Y$.

- $f$ is surjective $\iff$ $f$ has a right inverse ($f \circ g = \text{Id}_Y$).

- $f$ is injective $\iff$ $f$ has a left inverse ($g \circ f = \text{Id}_X$).

(This $g$ is not unique in general, cf. above ex.: $\pm \sqrt{y}$.)
Let \( f: X \to Y \) be a function.

Definition (1) For every \( A \subseteq X \) its **image** (under \( f \)) is the subset \( f(A) \subseteq Y \) defined by

\[
 f(A) = \{ y \in Y : \exists a \in A \text{ such that } y = f(a) \} = \{ f(a) : a \in A \}. 
\]

(2) For every \( C \subseteq Y \) its **inverse image** (via \( f \)) is the subset \( f^{-1}(C) \subseteq X \) given by

\[
 f^{-1}(C) = \{ x \in X : f(x) \in C \}. 
\]

Note: \( f^{-1}(C) \) makes sense even if \( f \) is not bijective (i.e., there's no \( f^{-1} \)).
- When \( f \) is bijective, \( f^{-1}(C) \) equals the *image of \( C \) under \( f^{-1} \)* (so no conflict in notation).

Image:

\[ X \xrightarrow{f} Y \]

\( A \rightarrow f(A) \)