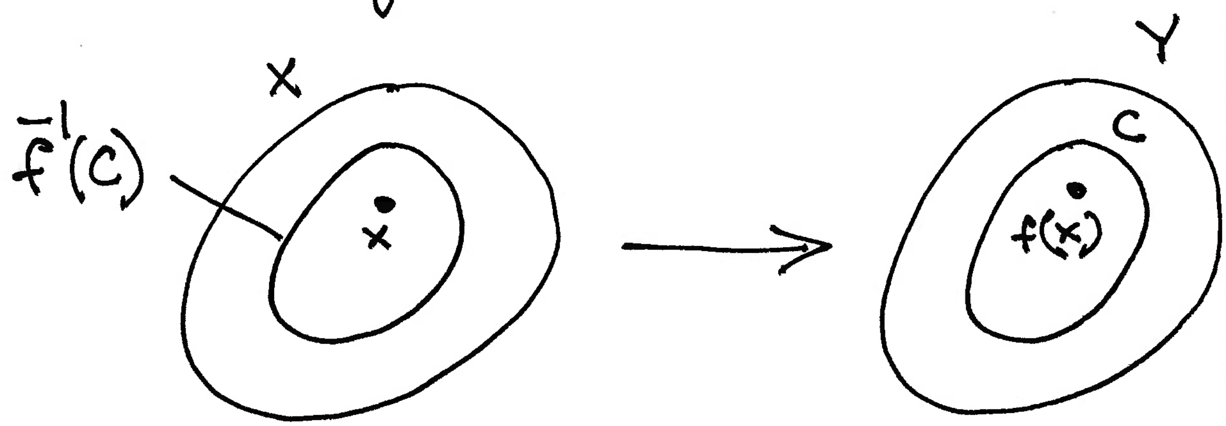
LECTURE 16

(Fri. FEB. 14, 2020)

Inverse Image:



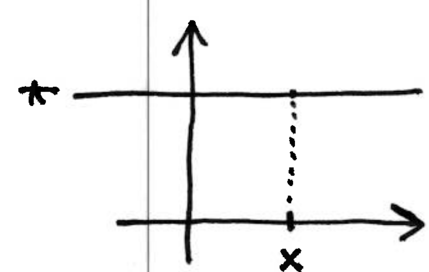
Special case: $f(X)$ is called the range of f.

$(f^{-1}(Y) = X)$ — Being "surjective" means $f(X) = Y$.

Ex. Let $f : \mathbb{R} \to \mathbb{R}$ be the constant function $f(x) = \pi$ for all x.

$\forall$ non-empty $A \subseteq \mathbb{R}$,

$$f(A) = \{\pi\}.$$



$\forall$ subset $C \subseteq \mathbb{R}$,

$$f^{-1}(C) = \begin{cases} \mathbb{R} & , \pi \in C \\ \emptyset & , \pi \notin C \end{cases}$$

Ex. Define $f : X \to Y$ by

$X = \{1, 2, 3, 4\}$

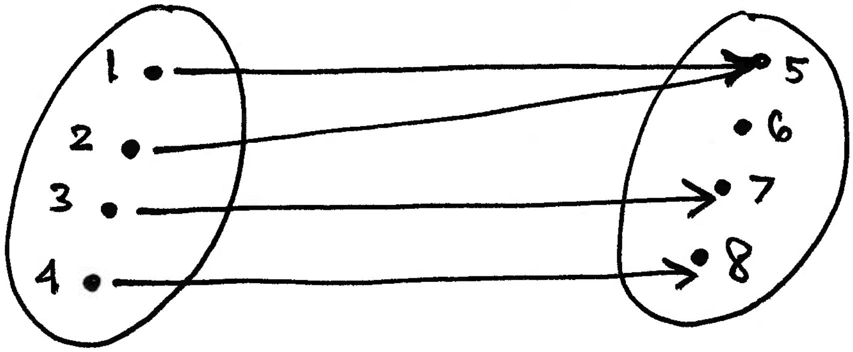
$Y = \{5, 6, 7, 8\}$

$1 \mapsto 5$

$2 \mapsto 5$

$3 \mapsto 7$

$4 \mapsto 8$



(not injective, nor surjective)

<u>range</u> $f(X) = \{5, 7, 8\}$.

$f^{-1}(\{5\}) = \{1, 2\}$.

$f^{-1}(\{6\}) = \emptyset$

$f^{-1}(\{7\}) = \{3\}$

$f^{-1}(\{8\}) = \{4\}$.

— other ex.,

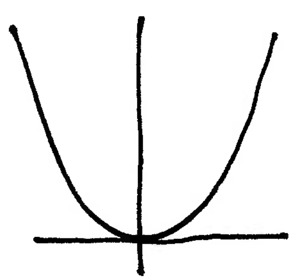
$f(\{1, 2\}) = \{5\}$.

etc.

<u>EX</u> $f: \mathbb{R} \to \mathbb{R}$, "<u>range</u>" is $f(\mathbb{R}) = [0, \infty)$
$f(x) = x^2$

$f([-1, 2]) = [0, 4]$.

$f^{-1}([0, 4]) = [-2, 2]$.



(not inj., nor surj.)

<u>Note</u>: In general $f(f^{-1}(C)) \subseteq C$,
— but this may <u>not</u> be an equality. ($f =$ constant, ex.)
<u>unless</u> f is <u>surjective</u>. — or above $f^{-1}(\{6\}) = \emptyset$

[If f surjective: Any $c \in C$ is of the form $c = f(x)$ for some $x \in X$ — necessarily in $f^{-1}(C)$.]

- Similarly $A \subseteq f^{-1}(f(A))$, but this may <u>not</u> be equality ($f =$ constant, ex.) <u>unless</u> f <u>injective</u>.

- above $f(\{1\}) = \{5\}$ & $A \neq \mathbb{R}$ non-empty.

[If f injective: If x belongs to the RHS, $f(x) \in f(A)$
I.e., $f(x) = f(a)$ for some $a \in A$.
$\Rightarrow$ $x = a$ lies in A.]

- Same flavor:

<u>Theorem</u> i) $f(A \cup B) = f(A) \cup f(B)$

ii) $f(A \cap B) \subseteq f(A) \cap f(B)$

[also true for <u>indexed</u> collections A_α, $\alpha \in I$.]

PROOF ii): $y \in f(A \cap B)$ is of the form $y = f(x)$ with $x \in A \cap B$. Then $x \in A$ and $x \in B$, so $y \in f(A)$ and $y \in f(B)$ ✓

The $\subseteq$ in ii) may be <u>strict</u>:

$f =$ constant ($= \pi$) and A, B <u>disjoint</u> $\neq \emptyset$.

$\emptyset \subset \{\pi\}$.

or $A = \{1\}$
$B = \{2\}$

It's an equality if f is <u>injective</u>:

Say $y = f(a) = f(b)$, where $a \in A$ and $b \in B$. Then $a = b \in A \cap B$ and $y = f(a) \in f(A \cap B)$ ✓

– For <u>inverse</u> images:

<u>Theorem</u> i) $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$

ii) $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$

[same for $\bigcup_{\alpha \in I} A_\alpha$ etc.] <u>always</u> <u>equality</u>.

PROOF ii) $x \in f^{-1}(C \cap D) \Leftrightarrow f(x) \in C \cap D$

$\Leftrightarrow f(x) \in C \wedge f(x) \in D$

$\Leftrightarrow x \in f^{-1}(C) \wedge x \in f^{-1}(D)$

$\Leftrightarrow x \in f^{-1}(C) \cap f^{-1}(D)$ ✓

<u>EX</u> A $m \times n$–matrix, $f: \mathbb{R}^n \to \mathbb{R}^m$ lin. transformation.

$f(x) = Ax$.

"<u>range</u>" is the image

$$im(f) = \{Ax : x \in \mathbb{R}^n\}$$

(= column space)

the kernel is the <u>inverse</u> <u>image</u> of $\vec{0} \in \mathbb{R}^m$:

$$ker(f) = f^{-1}(0) = \{x \in \mathbb{R}^n : Ax = 0\}$$

(= null space)

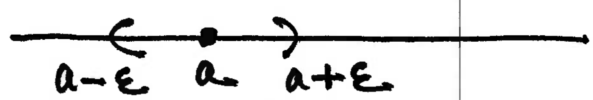
Ex. An "<u>open</u>" set in $\mathbb{R}$ is a union of open intervals.

arbitrary

I.e., $A \subseteq \mathbb{R}$ is <u>open</u> if $\forall a \in A \ \exists \varepsilon > 0$:

$$(a-\varepsilon, a+\varepsilon) \subseteq A.$$

exc.



entirely <u>contained in</u> A.

[If $\overline{A}$ is open we say A is "<u>closed</u>"].

— for instance $(0,\infty) \subseteq \mathbb{R}$ is open, $[0,\infty) \subseteq \mathbb{R}$ is closed.

$\{x\}$ is closed, not open.

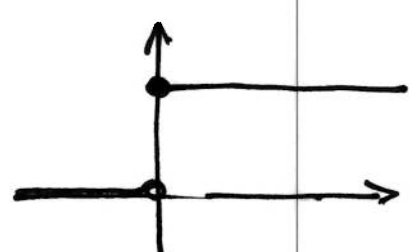
$f : \mathbb{R} \to \mathbb{R}$ function.

- $A \subseteq \mathbb{R}$ open $\overset{?}{\Longrightarrow}$ $f(A) \subseteq \mathbb{R}$ open — no, constant function $f(x) = \pi$ has $f(\mathbb{R}) = \{\pi\}$.
- $C \subseteq \mathbb{R}$ open $\overset{?}{\Longrightarrow}$ $f^{-1}(C) \subseteq \mathbb{R}$ open.

— no, consider the ex. $f(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0. \end{cases}$

$C = (0,\infty)$ has

$$f^{-1}(C) = \{x \in \mathbb{R} : f(x) > 0\}$$
$$= [0,\infty) \text{ <u>not</u> open.}$$



<u>Def</u> $f : \mathbb{R} \to \mathbb{R}$ is "<u>continuous</u>" if $f^{-1}(C)$ is open for every open $C \subseteq \mathbb{R}$.