LECTURE 17

(Wed. FEB. 19, 2020)

X finite, $X = \{x_1, x_2, \ldots, x_n\}$, $n = |X|$ its <u>cardinality</u>.

– Sending $1 \mapsto x_1, \ldots, n \mapsto x_n$ defines a <u>bijection</u> $f: \{1, 2, \ldots, n\} \to X$.

If Y finite, of the <u>same</u> cardinality, get a bijection

$$X \xrightarrow{f^{-1}} \{1, 2, \ldots, n\} \xrightarrow{g} Y.$$

Conversely, if $\exists$ bijection $X \to Y$, then $|X| = |Y|$.

(<u>no</u> bijection $\{1, 2, \ldots, m\} \to \{1, 2, \ldots, n\}$ when $m \neq n$)

– more generally/precisely:

<u>Theorem</u> X, Y finite sets.

(i) If there's an <u>injection</u> $f: X \to Y$, then $|X| \leq |Y|$.

(ii) If there's a <u>surjection</u> $f: X \to Y$, then $|X| \geq |Y|$.

Pf. (i) Let $Z = f(X)$ be its range. Then $X \to Z$, $x \mapsto f(x)$ is bijective. Therefore,

$$|X| = |Z| \leq |Y|, \text{ since } Z \subseteq Y.$$

(ii) If f is surjective it has a right inverse

(some $g: Y \to X$, $f \circ g = \mathrm{Id}_Y$).

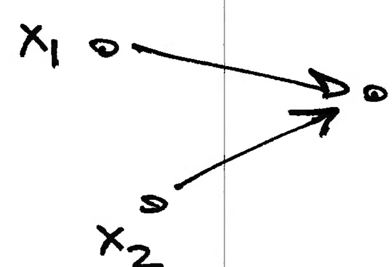
– Such a g is nec. injective, as it admits a left inverse — namely f.

By (i) applied to this g, $|Y| \leq |X|$. $\square$

Consequence of (i): If X, Y are finite, $|X| > |Y|$, then every function $f: X \to Y$ fails to be injective. I.e.,

$$\exists\, x_1 \neq x_2 \text{ in } X \text{ s.t. } f(x_1) = f(x_2).$$

This is known as the "pigeonhole principle"



$X = \{\text{pigeons}\}$

$Y = \{\text{holes}\}$.

If "more pigeons than holes, two pigeons (at least) fly into the same hole".

Ex Given $n+1$ (distinct) integers

$$a_0, a_1, \ldots, a_n$$

at least two of them are congruent mod n.

I.e., $\exists\, i \neq j$: $\quad a_i \equiv a_j \pmod n$

∘ Why? $X = \{a_0, a_1, \dots, a_n\} \subseteq \mathbb{Z}$, $|X| = n+1$.

$Y = \mathbb{Z}_n$, $|Y| = n$.

$f: X \to Y$ not injective, so $\exists\, a, b \in X$:

$a \mapsto [a]$.

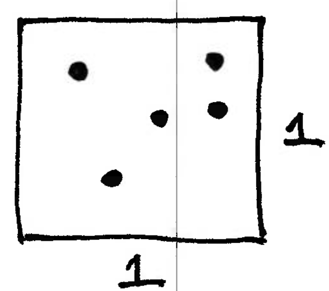
$[a] = [b]$

i.e., $a \equiv b \pmod{n}$.

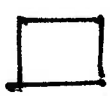
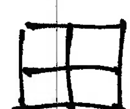
Ex. Given 5 points in a unit square: x_1, x_2, x_3, x_4, x_5

(possibly on boundary)

– At least two are within a distance $\frac{1}{\sqrt{2}}$ of each other:



$\exists\, i \neq j$: $\|x_i - x_j\| \leq \frac{1}{\sqrt{2}}$.

Why? Divide  into 4 smaller squares 

diameter $\sqrt{2}$ — diameter $\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$.

$X = \{x_1, \dots, x_5\}$

$Y = \{\text{smaller squares}\}$

Two lie in the same smaller square.

– Similarly: 9 points on a unit cube:

$\|x_i - x_j\| \leq \frac{\sqrt{3}}{2}$.

Ex $S = \{1, 2, \dots, n\}$.

F collection of subsets $A \subseteq S$,

with property: $X \cap Y \neq \emptyset$ for any two $X, Y \in F$.

(for instance take F to be all subsets $Z \cup \{n\}$ where $Z \subseteq \{1, 2, \dots, n-1\}$ varies)

CLAIM: For any such collection F, $|F| \leq 2^{n-1}$.

Partition $\mathcal{P}(S) = \bigcup \{A, \overline{A}\}$ by "pairing" A and $\overline{A}$.

has 2^{n-1} boxes.

If $|F| > 2^{n-1}$, some box contains two $X, Y \in F$.

i.e, $X = A$ and $Y = \overline{A}$, or vice versa.

But then $X \cap Y = \emptyset$.

App.: If $|X|=|Y|$ finite, $f: X \to Y$ is injective $\Longleftrightarrow$ surjective $\Longleftrightarrow$ bijective.

Why? If f injective, $Z = f(X)$ range.

Then $X \longrightarrow Z$ bijective. $|X| = |Z|$
$x \longmapsto f(x)$

$Z \subseteq Y$. Must have $Z = Y$.

same # elements.