LECTURE 18

(Fri. FEB. 21, 2020)

/ original context.

<u>Ex</u> (Dirichlet) Let $\alpha \in \mathbb{R}^*$ and $N \in \mathbb{N}$ be arbitrary.

- Form: $\{1\alpha, 2\alpha, 3\alpha, \ldots, N\alpha\}$.

At least <u>one</u> of them is $< \frac{1}{N}$ away from an integer.

I.e., $\exists n \in \{1, 2, \ldots, N\}$ and $\exists m \in \mathbb{Z}$:

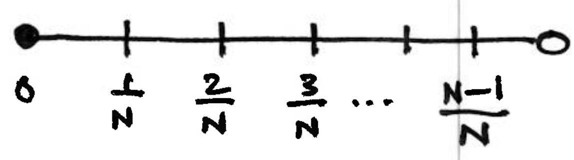
$$|n\alpha - m| < \frac{1}{N}.$$

- Why? Introduce $x_n = n\alpha - [n\alpha]$

<u>Note</u>: $0 \leqslant x_n < 1$ for all n. floor.

Partition $[0,1) = [0, \frac{1}{N}) \cup [\frac{1}{N}, \frac{2}{N}) \cup \cdots \cup [\frac{N-1}{N}, 1)$.

Has N subintervals.



$N+1$ numbers:

$x_0, x_1, \ldots, x_N$.

Two in the same subinterval, so $\exists\, r > s$ (wlog) in $\{0, 1, \ldots, N\}$ s.t.

$$|x_r - x_s| < \frac{1}{N}.$$

But,

$$x_r - x_s = (r\alpha - [r\alpha]) - (s\alpha - [s\alpha])$$

$$= \underbrace{(r-s)}_{n \in \{1, \ldots, N\}}\alpha - \underbrace{([r\alpha] - [s\alpha])}_{m \in \mathbb{Z}}.$$

11. / possibly infinite.

- Two sets X, Y have the same cardinality if there's a bijection (or "numerically equivalent")

$$f : X \longrightarrow Y$$

(f usually not unique: If there's some bijection there are many! - can compose f with any permutation of X).

Notation: $|X| = |Y|$.

(potentially confusing since $|X|$ makes no sense in isolation)

Obs:

- $|X| = |X|$ (take $f = \mathrm{Id}_X$)
- $|X| = |Y| \Longrightarrow |Y| = |X|$ (f^{-1} is bijective)
- $|X| = |Y| \wedge |Y| = |Z| \Longrightarrow |X| = |Z|$

- recall: ($f \circ g$ bijective if f, g are)

Intuitively $|X| = |Y|$ means one can pair elements $x \in X$ and $y \in Y$.

say

$$X = \{ x_1 , x_2 , x_3 , \dots \}$$

$$\updownarrow \quad \updownarrow \quad \updownarrow \quad \dots$$

$$Y = \{ y_1 , y_2 , y_3 , \dots \}$$

list without repetition.

$f : X \to Y$, $f(x_n) = y_n$ is a bijection.

(diff. orderings give diff. bijection).

"denumerable"

Def. X is <u>countable</u> if it's finite or $|X| = |\mathbb{N}|$.

— means there's an exhaustive list (w.o. repetitions) $X = \{x_1, x_2, x_3, \ldots\}$.

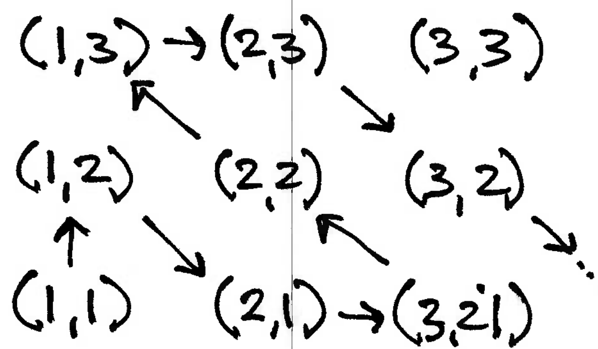
Ex 1) $\mathbb{Z}$ is <u>countable</u>.

explicit f: see p. 281.

$$\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \ldots\}$$
$$\updownarrow \ \updownarrow \ \updownarrow \ \updownarrow \ \ldots$$
$$\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, \ldots\}$$

2) $\mathbb{N} \times \mathbb{N}$ is <u>countable</u>.

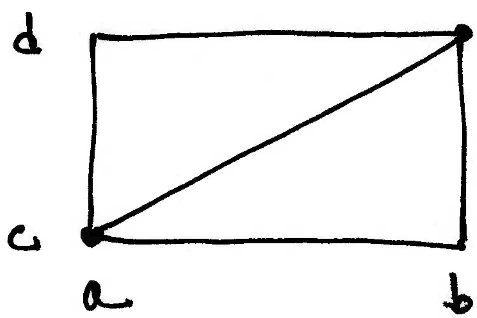
$$\mathbb{N} \times \mathbb{N} = \{(1,1), (1,2), (2,1), \ldots\}.$$



<u>LATER</u>: $\mathcal{P}(\mathbb{N})$ <u>uncountable</u>.

★ <u>Obs</u>: <u>Any</u> two open intervals (a,b) and (c,d) have $|(a,b)| = |(c,d)|$.

Why?



$$f(x) = \frac{d-c}{b-a}(x-a) + c$$

bijection.