LECTURE 19

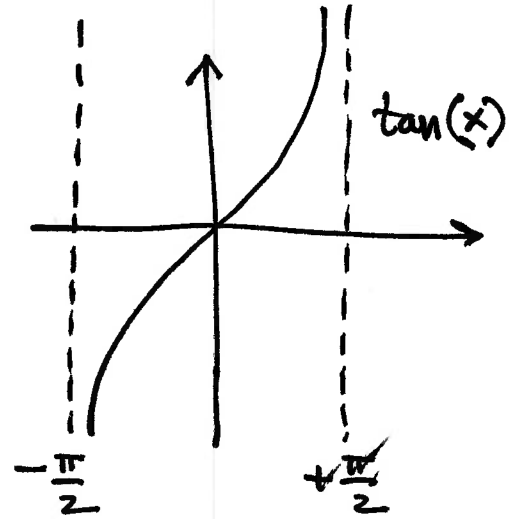
(Wed. FEB. 26, 2020)

EX $\tan : \left(-\frac{\pi}{2}, +\frac{\pi}{2}\right) \longrightarrow \mathbb{R}$ bijection.

Shows:

$$|\mathbb{R}| = \left|\left(-\frac{\pi}{2}, +\frac{\pi}{2}\right)\right| = |(a,b)|$$

$\forall a,b \in \mathbb{R}$.



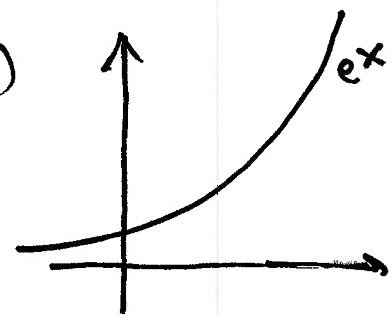
($\tan^{-1} = \arctan$)

($\exp^{-1} = \ln$)

EX $\exp : \mathbb{R} \longrightarrow (0,\infty)$ bijection.

Shows

$$|\mathbb{R}| = |(0,\infty)|.$$



(perhaps unbounded)

$\longrightarrow$ So ANY open interval is num. equiv. to $\mathbb{R}$.

Theorem X any set. Then X is never numerically equivalent to its power set $\mathcal{P}(X)$.

In fact, there's no surjection

$$f : X \longrightarrow \mathcal{P}(X).$$

[In particular $\mathcal{P}(\mathbb{N})$ is uncountable.]

Obvious if X finite:

$n = |X|$

$2^n = |\mathcal{P}(X)| > n.$

PROOF. Suppose we have a surjection

$$f : X \longrightarrow \mathcal{P}(X). \qquad x \longmapsto A_x.$$

Notation: Let $A_x \subseteq X$ be the subset $f(x)$ associated with $x \in X$ via f.

- Consider the subset $B \subseteq X$ defined by:

$$B = \{ x \in X : \; x \notin A_x \}.$$

Since f is surjective, $B = A_t$ for some $t \in X$.

Does A_t contain t?

- YES: If $t \in A_t$, deduce $t \notin B$ (by def. of B). This says $t \notin A_t$, contradiction.
- NO: If $t \notin A_t$, deduce $t \in B$. Says $t \in A_t$, contradiction.

Either way, a contradiction. So no surjection exists. $\square$

Cantor's diagonal argument: $(0,1)$ uncountable.

$x \in (0,1)$ has a binary expansion $x = 0.c_1 c_2 c_3 \ldots\ldots$ all $c_i \in \{0,1\}$.

Ex. $\frac{1}{8} = 0.00100\ldots\ldots = 2^{-3}$.

- Suppose we have a complete list of all $x \in (0,1)$ (w.o. repetitions):

- subtlety: Assume exp. doesn't end in 1's.

  ex.

  $$\frac{1}{2} = 0.10000\ldots\ldots$$
  $$= 0.01111\ldots\ldots$$

$x_1 = 0.\underline{1}0101\ldots$

$x_2 = 0.0\underline{1}110\ldots$

$x_3 = 0.10\underline{1}10\ldots$

$\vdots$

$x = 0.0001\ldots$

Look at the <u>diagonal</u> <u>digits</u>.

– Form the number x whose n^{th} digit is the "<u>opposite</u>" of the n^{th} digit of x_n.

Then $x \neq x_n$ for any n, therefore <u>not</u> on the list.

<u>FACT</u>: $|(0,1)| = |\mathcal{P}(\mathbb{N})|$. (cf. p. 299)

(requires more work)