LECTURE 2
(Wed. JAN. 8, 2020)
Def. If $A \subseteq \mathbb{X}$ its complement ($\overline{A}$) is the family of all elements of $\mathbb{X}$ not in $A$:

$$\overline{A} = \mathbb{X} - A = \{ x \in \mathbb{X} : x \notin A \}.$$  

(sometimes $A^c$)

Obs: $\overline{\overline{A}} = A$.

Ex. ($\mathbb{X} = \mathbb{R}$)

$$\overline{\mathbb{Q}} = \{ x \in \mathbb{R} : x \text{ not rational} \}$$

"rational numbers". \[ \text{II. } \pi \notin \overline{\mathbb{Q}}. \]

Def. The "power set" of $\mathbb{X}$ is the collection of all its subsets:

$$\mathcal{P}(\mathbb{X}) = \{ A : A \subseteq \mathbb{X} \}.$$ (never empty)

Thus, $\emptyset \in \mathcal{P}(\mathbb{X})$ and $\mathbb{X} \in \mathcal{P}(\mathbb{X})$.

Ex. $\mathbb{X} = \{1, 2, 3, 7\}$.

$$\mathcal{P}(\emptyset) = \{\emptyset\}.$$

$$\mathcal{P}(\mathbb{X}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{7\}, \{1, 2\}, \{1, 3\}, \{1, 7\}, \{2, 3\}, \{2, 7\}, \{3, 7\}, \{1, 2, 3\}, \{1, 2, 7\}, \{1, 3, 7\}, \{2, 3, 7\}, \mathbb{X}\}.$$  

In general, if $\mathbb{X}$ is finite with $|\mathbb{X}| = n$, then $|\mathcal{P}(\mathbb{X})| = 2^n$ (exc.)
$X = \{x_1, x_2, \ldots, x_n\}$, how many subsets? \\
A \subseteq X \\
include $x_1$? yes/no. \\
include $x_n$? yes/no. \\
Combinatorics: $2^n$ possibilities. \\
\[|\mathcal{P}(X)| = 2^n.\]
\[\text{power set -- consisting of all subsets } A \subseteq X.\]

Set operations: Fix a set $X$ ("universal") \\
$A$ and $B$ two subsets of $X$. \\
\[A \cap B = \{x: x \in A \text{ and } x \in B\}\]
\[A \cup B = \{x: x \in A \text{ or } x \in B\}\]
Def. The intersection is \\
The union is \\
\[A \cap \emptyset = \emptyset\]
\[A \cup \emptyset = A\]
\[A \cap X = A\]
\[A \cup X = X.\]

\[A, B \text{ possibly in both.} \]

Terminology: When $A \cap B = \emptyset$, we say $A, B$ are disjoint.
\textbf{Ex. (x=Z)} \hspace{1cm} A = \{x \in \mathbb{Z} : x \geq 3 \} = \{3, 4, 5, \ldots \} \\
B = \{x \in \mathbb{Z} : x \leq 5 \} = \{\ldots, 3, 4, 5 \} \\
\text{Here, } A \cap B = \{3, 4, 5 \} \text{ and } A \cup B = \mathbb{Z}.

\textbf{Ex. } A \cup \overline{A} = \mathbb{X} \hspace{1cm} \text{Def. The difference is } A - B = \{x \in A : x \notin B \} = A \cap \overline{B}

\text{Useful observation: For any two } A, B \subseteq \mathbb{X},

A \cup B = (A - B) \cup (B - A) \cup (A \cap B) \text{ all mutually disjoint.}

[Unions/intersections of three (or more) subsets are def. in the obvious way. I’ll move later..]
Theorem (De Morgan's laws) \( A, B \subseteq X \).

(i) \( \overline{A \cup B} = \overline{A} \cap \overline{B} \)

(ii) \( \overline{A \cap B} = \overline{A} \cup \overline{B} \)

Why? An \( x \in X \) lies in the LHS precisely when

(i) \( x \) is not in \( A \cup B \), i.e. when \( x \) is not in \( A \) and not in \( B \). This means \( x \in \overline{A} \) and \( x \in \overline{B} \); equivalently \( x \) lies in the RHS.

We can now deduce (ii) by applying (i) to \( \overline{A} \) and \( \overline{B} \):

\[
\overline{\overline{A} \cup \overline{B}} = \overline{\overline{A}} \cap \overline{\overline{B}} = \overline{A} \cap \overline{B}.
\]

Taking the complement:

\[
\overline{A \cup B} = \overline{A \cap B}.
\]

which is (ii).