LECTURE 20

(Fri. FEB. 28, 2020)

/ finite or $|X| = |\mathbb{N}|$

<u>Theorem</u> X countable. Then every $A \subseteq X$ is countable.

PROOF. May assume A infinite (then X also infinite)

List X:

$$X = \{x_1, x_2, x_3, \ldots\}. \qquad \begin{array}{c} \mathbb{N} \xrightarrow{\text{bij.}} X \\ n \longmapsto x_n \end{array}$$

Since $A \neq \emptyset$ it contains some of the x_n's.
(in fact ∞ many)

Let

$$S = \{n \in \mathbb{N} : x_n \in A\}$$

&

$$m = \min(S).$$

— this is a non-empty subset of $\mathbb{N}$, so it has a <u>smallest element</u>.
("Well-ordering principle")

(In other words x_m is the "first" element of X belonging to A.) — rename it:

$$a_1 = x_m.$$

Continue this way: Let a_2 be the "first" element of X belonging to $A - \{a_1\}$, etc.

– This gives an ∞ sequence of distinct elements of A:

$$a_1, a_2, a_3, \ldots\ldots$$

Claim this list is <u>exhaustive</u> (i.e. all of A).

– Recall the "remaining":

$$\begin{array}{ccc} a_1 & a_2 & a_3 \\ \| & \| & \| \\ x_{m_1} & x_{m_2} & x_{m_3} \end{array}$$

(where $m_1 < m_2 < m_3 < \cdots$)

Consider an abitrary $a \in A$ (show it's on the list)
Certainly $a = x_n$ for some $n \in \mathbb{N}$.

- If n is one of the m_i's we're done (then $a = x_{m_i} = a_i$).
- If not, there's a unique i s.t. $m_i < n < m_{i+1}$.

But by definition, m_{i+1} is the smallest index r for which $x_r \in A - \{a_i\}$

However $n < m_{i+1}$ and $x_n = a \in A - \{a_i\}$

Contradiction. $\square$

($a \neq a_i$? Otherwise $\begin{array}{ccc} a & = & a_i \\ \| & & \| \\ x_n & & x_{m_i} \end{array}$ $\Rightarrow n = m_i$)

Corollary X any set.
If X has an uncountable subset, X itself is uncountable.

– in particular $\mathbb{R}$ is uncountable since we know $(0,1) \subseteq \mathbb{R}$ is uncountable.

Another application:

Corollary $\mathbb{Q}$ is countable.

Why? Know $\mathbb{N}$ and $\mathbb{Z}$ are countable, therefore so is $\mathbb{Z}\times\mathbb{N}$.
The function

$$f: \mathbb{Z}\times\mathbb{N} \longrightarrow \mathbb{Q}$$
$$(a,b) \longmapsto \frac{a}{b}$$

is surjective (but not injective: $f(1,1) = 1 = f(2,2)$)
$\hookrightarrow$ admits a right inverse $g: \mathbb{Q} \to \mathbb{Z}\times\mathbb{N}$, which
$(f\circ g = \mathrm{Id}_{\mathbb{Q}})$ is necessarily injective, since it admits a left inverse (f, ex.)

Thus $\mathbb{Q} \xrightarrow{g} \mathbb{Z}\times\mathbb{N}$
gives bijection
$$\mathbb{Q} \longrightarrow g(\mathbb{Q})$$
$$r \longmapsto g(r)$$

which shows $|\mathbb{Q}| = |g(\mathbb{Q})| = |\mathbb{N}|$

uses Theorem: $g(\mathbb{Q}) \subseteq \mathbb{Z}\times\mathbb{N}$ is countable. $\square$

Remark: $A, B \subseteq X$, both countable subsets. Then so is $A\cup B$.

- First, if $A\cap B = \emptyset$:
$$A = \{a_1, a_2, \ldots\}$$
$$B = \{b_1, b_2, \ldots\}$$
$$A\cup B = \{a_1, b_1, a_2, b_2, \ldots\}.$$

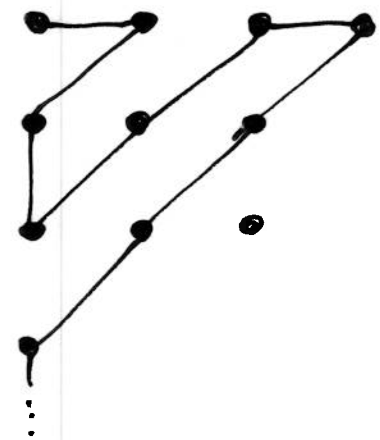
- Can reduce to the disjoint case by observing: $A\cup B = (A-B)\cup(B-A)\cup(A\cap B)$
(countable? eg. $A-B\subseteq A$ etc.) pw. disjoint countable sets.

More generally, supp. we have a sequence of subsets: countable $\subseteq X$

$$A_1, A_2, A_3, \ldots, A_n, \ldots$$

Then $\bigcup_{n=1}^{\infty} A_n$ is also countable

"usual snake"



(why? Lists $A_1 = \{a_{11}, a_{12}, \ldots\}$

$A_2 = \{a_{21}, a_{22}, \ldots\}$

$\vdots$

gives $f: \mathbb{N}\times\mathbb{N} \longrightarrow X$ whose range is the union.

$(m,n) \longmapsto a_{mn}$

may not be injective if "overlaps", ex. $A_1 \cap A_2 \neq \emptyset$.

Choosing a left inverse shows $\left|\bigcup_{n=1}^{\infty} A_n\right| = |\mathbb{N}|$ since $\mathbb{N}\times\mathbb{N}$ is countable.)

(of $\mathbb{N}\times\mathbb{N} \to \bigcup_{n=1}^{\infty} A_n$)

- NOTE:

Conclusion obviously false for arbitrary indexed unions $\bigcup_{\alpha\in I} A_\alpha$ of countable sets A_α.

EX. $I = \mathbb{R}$, $A_\alpha = \{\alpha\}$, then $\bigcup_{\alpha\in\mathbb{R}} \{\alpha\} = \mathbb{R}$.

finite. uncountable.

(I must be countable.)