LECTURE 20
(Fri. FEB. 28, 2020)
Theorem 1: If \( |X| = \aleph_0 \) (i.e., \( X \) is countable) and \( A \subseteq X \) is countable, then every \( A \subseteq X \) is countable.

Proof: Let \( A \neq \emptyset \) and \( \aleph_0 \) (then \( X \) also infinite).

List \( X \):

\[
X = \{ x_1, x_2, x_3, \ldots \}
\]

Since \( A \neq \emptyset \), it contains some of the \( x_n \)'s. (In fact \( \aleph_0 \) many)

Let \( S = \{ n \in \mathbb{N} : x_n \in A \} \) — this is a non-empty subset of \( \mathbb{N} \), so it has a smallest element.

(“Well-ordering principle”)

\( \min(S) = m \).

(“First” element of \( X \) belonging to \( A \).) Rename it:

\[
a_1 = x_m.
\]

Continue this way: Let \( a_2 \) be the “first” element of \( X \) belonging to \( A \) — \( \{a_1, a_2, \ldots \} \).

This gives an \( \aleph_0 \) sequence of distinct elements of \( A \):

\[
a_1, a_2, a_3, \ldots
\]

Claim this list is exhaustive (i.e., all of \( A \)).
Recall the "remaining": \( a_1 \quad a_2 \quad a_3 \)
\[ \equiv \quad \equiv \quad \equiv \]
(where \( m_1 < m_2 < m_3 < \ldots \) \( x_{m_1} \quad x_{m_2} \quad x_{m_3} \))

Consider an arbitrary \( a \in A \) (show it's on the list).
Certainly \( a = x_n \) for some \( n \in \mathbb{N} \).

If \( n \) is one of the \( m_i \)'s we've done (then \( a = x_{m_i} = a_i \)).

If not, there's a unique \( i \) s.t. \( m_i < n < m_{i+1} \).

But by definition, \( m_{i+1} \) is the smallest index \( r \) for which \( x_r \in A - \{a_i\} \).
However \( n < m_{i+1} \) and \( x_n = a \in A - \{a_i\} \).

Contradiction. \( \square \)

\( a \neq a_i \) (otherwise \( a = a_i \))

\( \Rightarrow n = m_i \)

\( x_n \quad x_{m_i} \)

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**Corollary 1**: any set.
If \( X \) has an uncountable subset, \( X \) itself is uncountable.

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**in particular** \( \mathbb{R} \) is uncountable, since we know \( (0, 1) \subseteq \mathbb{R} \) is uncountable.

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**Another application:**

**Corollary 2**: \( \mathbb{Q} \) is countable.
Why? Know $\mathbb{N}$ and $\mathbb{Z}$ are countable, therefore so is $\mathbb{Z} \times \mathbb{N}$.

The function

$$f: \mathbb{Z} \times \mathbb{N} \rightarrow \mathbb{Q}$$

$$(a, b) \mapsto \frac{a}{b}$$

is surjective (but not injective: $f(1,1) = 1 = f(2,2)$)

$\Rightarrow$ admits a right inverse $g: \mathbb{Q} \rightarrow \mathbb{Z} \times \mathbb{N}$, which

$$(f \circ g = \text{Id}_\mathbb{Q})$$

is necessarily injective.

Thus $\mathbb{Q} \overset{g}{\rightarrow} \mathbb{Z} \times \mathbb{N}$

gives bijection $\mathbb{Q} \overset{g(\cdot)}{\rightarrow} g(\mathbb{Q})$

$r \mapsto g(r)$

which shows $|\mathbb{Q}| = |g(\mathbb{Q})| = |\mathbb{N}|$

use Theorem:

$g(\mathbb{Q}) \subseteq \mathbb{Z} \times \mathbb{N}$ is countable. □

Remark: $A, B \subseteq X$, both countable subsets. Then $\infty$ is $A \cup B$.

- First, if $A \cap B = \emptyset$:

  $A = \{a_1, a_2, \ldots \}$

  $B = \{b_1, b_2, \ldots \}$

  $A \cup B = \{a_1, b_1, a_2, b_2, \ldots \}$.

- Can reduce to the

  disjoint case by observing: $A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$

  (countable, e.g. $A - B \subseteq A$ etc.) with disjoint countable sets.
More generally, suppose we have a sequence of subsets:

\[ A_1, A_2, A_3, \ldots, A_n, \ldots \]

Then \( \bigcup_{n=1}^{\infty} A_n \) is also countable.

(Why? List: \( A_1 = \{a_{11}, a_{12}, \ldots\} \)?
\[ A_2 = \{a_{21}, a_{22}, \ldots\} \]
\[ \vdots \]

This gives a function \( f: \mathbb{N} \times \mathbb{N} \to X \) whose range is the union. \( f(m,n) = a_{mn} \).

Choosing a left inverse shows \( \left| \bigcup_{n=1}^{\infty} A_n \right| = |\mathbb{N}| \) since \( N \times N \) is countable.

**Note:** Conclusion obviously false for arbitrary indexed unions \( \bigcup_{\alpha \in I} A_{\alpha} \) of countable sets \( A_{\alpha} \).

**Ex:** \( I = \mathbb{R} \), \( A_{\alpha} = \{x^{\alpha}\} \), then \( \bigcup_{\alpha \in \mathbb{R}} \{x^{\alpha}\} = \mathbb{R} \).

(I must be countable.)