LECTURE 21
(Mon. MAR. 2, 2020)

Remarks/Ex:

(1) Algebraic numbers = roots of polynomial equations w. $\mathbb{Q}$-coeffs.

~ I.e., number x is "algebraic" if satisfies relation

(monic, $a_n = 1$.) $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$, $n \geq 1$.

where all $a_i \in \mathbb{Q}$.

EX $\sqrt{2}$, $\sqrt{2}+\sqrt{3}$, $\sqrt{2+\sqrt{3}}$ ... (exc – check this).

Thm. The set of algebraic numbers is countable.

Idea: The set of monic $\mathbb{Q}$-polynomials $= X$.

$$X = \bigcup_{n=1}^{\infty} X_n \quad , \quad X_n \text{ those of degree } n.$$

Note $\mathbb{Q}^n = \underbrace{\mathbb{Q} \times \cdots \times \mathbb{Q}}_{n} \longrightarrow X_n$ is a bijection.

$$(a_0, \ldots, a_{n-1}) \longmapsto x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

Since $\mathbb{Q}$ is countable, so is $\mathbb{Q}^n$ and therefore X_n, and X. (being a countable union)

Fact: Each poly has $\leq n$ roots.

$X = \{f_1, f_2, f_3, \ldots\}$

roots

$x_{11}, \ldots, x_{1n}$ $\quad$ $x_{21}, \ldots, x_{2n}$ $\quad$ $x_{31}, \ldots, x_{3n}$

~ Complete list of alg. numbers ✓

$\Rightarrow$ The set of <u>non</u>-algebraic ("transcendental") numbers is <u>uncountable</u>.

– so they <u>exist</u>! (in fact the majority). (since $\mathbb{R}$ is)

<u>EX</u>: π, e, Liouville number $\sum_{n=1}^{\infty} 10^{-n!}$ (non-trivial!).

(2) <u>Question</u>: $|(0,1)| \overset{?}{=} |[0,1]|$ (i.e., is there a <u>bijection</u> $f: [0,1] \to (0,1)$? – can't be <u>continuous</u>!)

$\star$) <u>Schröder</u> – <u>Bernstein</u> (p. 298):

$|X| \leq |Y| \wedge |Y| \leq |X| \Rightarrow |X| = |Y|$.

Omit proof. Highly non-trivial!

Boils down to: | Suppose $Y \subseteq X$ and there's an injection $X \to Y$. Then there's a <u>bijection</u> $X \to Y$.

– Apply: $(0,1) \subseteq [0,1]$ so $|(0,1)| \leq |[0,1]|$.

$$\begin{array}{c} |(0,1)| \\ \| \\ |\mathbb{R}|. \end{array}$$ – know (use tan etc.)

$[0,1] \subseteq \mathbb{R}$ so $|[0,1]| \leq |\mathbb{R}|$.

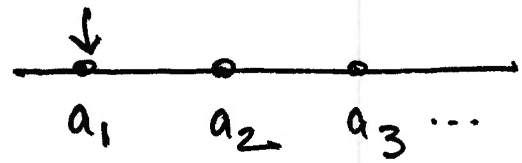
Thm (S.–B.) <u>implies</u> $|\mathbb{R}| = |(0,1)| = |[0,1]|$.

$\star$)

Induction (§6). $\mathbb{N} = \{1, 2, 3, \dots\}$

Recall "well-ordering principle" says that

any non-empty subset $A \subseteq \mathbb{N}$ has a smallest element. (*)



i.e. $\exists s \in A$ such that $s \leq a$ holds for all $a \in A$.

- Such an s is necessarily unique: If $t \in A$ had the same property, $s \leq t$ and $t \leq s$. I.e., $s = t$.

⤳ "THE" smallest element of A.

Note: Property (*) fails for (some) subsets of $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{Q}$.

EX: ∘ $[0, \infty)$ has a smallest element $s = 0$, but $(0, \infty)$ doesn't.

∘ $A = \{\text{neg. integers}\}$ has no smallest s. (or $A = \mathbb{Z}$)

∘ $A = \mathbb{Q} \cap (0, \infty)$ ——— " ———
"pos. rationals"

$\mathbb{N}$:

$A = \{n \in \mathbb{N} : n^2 > 5\} = \{3, 4, 5, \dots\}$, $s = 3$.

$B = \{n \in \mathbb{N} : 5 \mid n\} = \{5, 10, 15, \dots\}$, $s = 5$.

The Principle of Mathematical Induction:

$P(n)$: math. statement depending on $n \in \mathbb{N}$.

Suppose (1) $P(1)$ true, and

(2) $P(n) \Longrightarrow P(n+1)$ holds for all $n \in \mathbb{N}$.

Then $P(n)$ is true for all $n \in \mathbb{N}$.

– Intuitively ("falling dominoes"):

$$P(1) \Longrightarrow P(2) \Longrightarrow P(3) \Longrightarrow P(4) \Longrightarrow \cdots$$

Formally: Suppose there's at least some $m \in \mathbb{N}$ for which $P(m)$ is false. I.e., the set

$$A = \{ m \in \mathbb{N} : P(m) \text{ false} \}$$

is non-empty. By well-ordering it has a smallest element $s \in A$. In other words

$$P(s) \text{ false} \;\wedge\; P(n) \text{ true for all } n < s.$$

Note: $s > 1$ since $1 \notin A$, we're assuming $P(1)$ true.

Therefore we may take $n = s - 1 < s$ above.

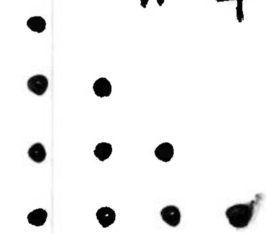
This contradicts (2): $\overset{\uparrow}{\mathbb{N}}$ $\qquad P(n) \Longrightarrow P(s)$. ✓

Examples

(1) Triangular numbers (cf. HW0):

$$T_n = 1+2+3+\cdots+n$$

n=4



n	T_n
1	1
2	3
3	6
4	10
5	15

Claim: "closed/explicit" formula,

$P(n)$: $T_n = \frac{1}{2}n(n+1)$, $\forall n \in \mathbb{N}$.

Proof by induction:

- Base step ($n=1$): Check $T_1 = \frac{1}{2}\cdot 1 \cdot (1+1)$
  - verifies $P(1)$ is true. (ok, both sides equal 1)

- Inductive step: Let $n \in \mathbb{N}$ be arbitrary and assume $P(n)$ holds, i.e. that

the "induction hypothesis".

$$T_n = \frac{1}{2}n(n+1).$$

- Use that to show $P(n+1)$ holds, i.e. that

$$T_{n+1} \underset{?}{=} \frac{1}{2}(n+1)(n+2).$$

- How? $T_{n+1} = T_n + (n+1) = \frac{1}{2}n\underline{(n+1)} + \underline{(n+1)}$

(factor out $n+1$)

$$= \frac{1}{2}(n+1)(n+2).$$ ✓ done.