LECTURE 22

(Wed. MAR. 4, 2020)

(Alternative: $T_n = 1 + 2 + 3 + \cdots + n$

$$T_n = n + (n-1) + (n-2) + \cdots + 1$$

$$2T_n = (n+1) + (n+1) + (n+1) + \cdots + (n+1)$$

$$= n(n+1).$$)

(2) Geometric sums: For any $x \in \mathbb{R}$, fixed, look at

$$s_n = 1 + x + x^2 + \cdots + x^n$$

Claim: $s_n = \dfrac{x^{n+1} - 1}{x-1}$, all $n \geq 0$, provided $x \neq 1$.

Proof by induction:

- Base step ($n=1$): Check $s_1 \stackrel{?}{=} \dfrac{x^{1+1} - 1}{x-1}$

  (both sides equal $x+1$).

  [Remark: $s_0 = 1 = \frac{x-1}{x-1}$ ok too.]

- Inductive step: $n \in \mathbb{N}$ arbitrary.

  Assume $s_n = \dfrac{x^{n+1} - 1}{x-1}$.

  Show $s_{n+1} = \dfrac{x^{n+2} - 1}{x-1}$.

– How? $s_{n+1} = s_n + x^{n+1} = \dfrac{x^{n+1} - 1}{x-1} + x^{n+1} =$

$$= \frac{x^{n+1}-1+x^{n+1}(x-1)}{x-1} = \frac{x^{n+1}-1+x^{n+2}-x^{n+1}}{x-1} = \frac{x^{n+2}-1}{x-1} \checkmark$$

cancel

(Alternative: $xs_n = x + x^2 + x^3 + \dots + x^{n+1}$
$s_n = 1 + x + x^2 + \dots + x^n$

terms cancel.

~ difference

$$xs_n - s_n = x^{n+1} - 1$$
$$\| $$
$$(x-1)s_n$$
)

*) Consequence: If $|x|<1$, then $s_n \longrightarrow \frac{1}{1-x}$.

(Exc: Use that $x^{n+1} \to 0$.)

~ For any real $x > -1$,

Ex ("Bernoulli's Inequality"): $(1+x)^n \geq 1+nx$, $\forall n \in \mathbb{N}$.

- Base step ($n=1$): $(1+x)^1 \geq 1+1x$ — equal $\checkmark$. ($n=0$ ok)

- Ind. step: $(1+x)^{n+1} = \underbrace{(1+x)^n}_{\geq 1+nx}\underbrace{(1+x)}_{\text{positive}} \geq (1+nx)(1+x)$

ok for n. assume. "ind. hypothesis".

$$(1+nx)(1+x) = \underbrace{1+x+nx}_{1+(n+1)x} + \underbrace{nx^2}_{\geq 0}.$$

~ altogether $\geq 1+(n+1)x$ $\checkmark$

∘ <u>Application</u>: "AM — GM Inequality".

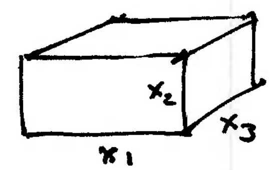
arithmetic mean:

$$A = \frac{1}{n}(x_1 + \cdots + x_n)$$

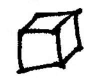
geometric mean:

$$G = \sqrt[n]{x_1 \cdots x_n}$$

[Here all $x_i > 0$.]



cube w. same volume?



side $= G$.

<u>Thm</u> $\boxed{A \geq G}$.

PF. Induction on n. $(n=1: \; A = x_1 = G)$.

Suppose true for $n-1$, $n>1$. Show it for n.

— reformulation,

∘ <u>BERNOULLI</u>:

$$t^n \geq 1 + n(t-1)$$

for $n \in \mathbb{N}$, $t > 0$.

— apply to $t = \frac{A_n}{A_{n-1}} > 0$.

$\cancel{\frac{A^n}{A_{n-1}}}$

Bernolli.

$$\left(\frac{A_n}{A_{n-1}}\right)^n \geq 1 + n\left(\frac{A_n}{A_{n-1}} - 1\right)$$

$$= \frac{A_{n-1} + nA_n - nA_{n-1}}{A_{n-1}}$$

$$= \frac{nA_n - (n-1)A_{n-1}}{A_{n-1}}$$

— By <u>induction</u>

$$= \frac{x_n}{A_{n-1}},$$

i.e. $A_n^n \geq x_n A_{n-1}^{n-1} \geq x_n (x_1 \cdots x_{n-1})$

$= G_n^n$ $\square$