LECTURE 23
(Fri. MAR. 6, 2020)

§6.3.

Strong Induction Principle: $P(n)$ math. statement indexed by $n \in \mathbb{N}$.

Suppose:

(1) $P(1)$ true.

(2) For all $n \in \mathbb{N}$,

$$P(1) \wedge P(2) \wedge \cdots \wedge P(n) \Longrightarrow P(n+1)$$

[i.e., if $P(m)$ true for all $m \leq n$, then $P(n+1)$ is true.].

Then: $P(n)$ is true for all $n \in \mathbb{N}$.

— follows from (regular) induction applied to

$$Q(n) = P(1) \wedge \cdots \wedge P(n).$$

Indeed $Q(1) = P(1)$ is true, and $Q(n) \Rightarrow Q(n+1)$.
Thus $Q(n)$ is true $\forall n \in \mathbb{N}$. Equivalently $P(n)$ true $\forall n \in \mathbb{N}$ ✓

"Strong" $\rightsquigarrow$ stronger induction hypothesis.
(not only is $P(n)$ assumed true, but all $P(1), P(2), \ldots, P(n)$)

Remark:
- well-ordering
- regular induction
- strong induction

$\Bigg\}$ all principles are equivalent.

Ex. strong induction $\Longrightarrow$ well-ordering.

Why? $A \subseteq \mathbb{N}$ non-empty. Suppose no smallest element.

I.e., $\forall s \in A \; \exists a \in A : a < s$.

Let $P(n): n \notin A$. Apply strong induction:

- Base step ($n=1$): $P(1)$ true — otherwise $1 \in A$ would be smallest.
- Ind. step: Suppose all $P(1), \ldots, P(n)$ are true, meaning $1 \notin A, \ldots, n \notin A$.
  Then $P(n+1)$ must be true, otherwise $n+1 \in A$ would be smallest. ✓

Conclude: $P(n)$ true $\forall n \in \mathbb{N}$.

— I.e., $n \notin A$ for all n.

This says $A = \emptyset$.

contradiction. $\square$

Ex (Fibonacci numbers) $F_1, F_2, F_3, \ldots$

— defined recursively: Initial conditions,

$$F_1 = F_2 = 1.$$

and for $n > 2$,

$$F_n = F_{n-1} + F_{n-2}.$$

n	F_n
1	1
2	1
3	2
4	3
5	5
6	8
7	13
8	21
$\vdots$	$\vdots$

Thm. (BINET'S FORMULA): $\forall n \in \mathbb{N}$,

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

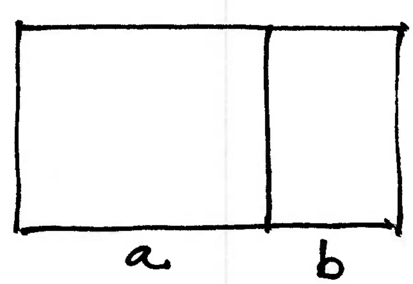
PROOF. By strong induction. First, introduce notation:

$$\begin{cases} \alpha = \dfrac{1+\sqrt{5}}{2} = 1.618\ldots \text{ (golden ratio)} \\ \beta = \dfrac{1-\sqrt{5}}{2} = -0.618\ldots \end{cases}$$

The two roots of $x^2 - x - 1 = 0$.

I.e., have relations:

$$\alpha^2 = \alpha + 1 \quad \text{and} \quad \beta^2 = \beta + 1.$$



$$\frac{a+b}{a} = \frac{a}{b} = \alpha.$$

∘ Base step ($n=1$):

$$1 = F_1 \overset{?}{=} \frac{1}{\sqrt{5}}(\alpha - \beta) = \frac{1}{\sqrt{5}} \cdot \sqrt{5}$$

ok.

- Induction step: Let $n \in \mathbb{N}$ and suppose Binet's formula holds for all indices $1, 2, \ldots, n$.

[strong ind. hypothesis].

— To show: $F_{n+1} \overset{?}{=} \frac{1}{\sqrt{5}}\left(\alpha^{n+1} - \beta^{n+1}\right)$.

From the definition of the (F_n):

$$F_{n+1} = F_n + F_{n-1} \qquad \left(\underset{n=1}{\text{convention}}: F_0 = 0\right)$$

Key point — since we're using strong induction, we can use Binet's formula for both F_n and F_{n-1}!

implies:

$$F_{n+1} = \frac{1}{\sqrt{5}}\left(\alpha^n - \beta^n\right) + \frac{1}{\sqrt{5}}\left(\alpha^{n-1} - \beta^{n-1}\right)$$

$$= \frac{1}{\sqrt{5}}\left(\alpha^n + \alpha^{n-1}\right) - \frac{1}{\sqrt{5}}\left(\beta^n + \beta^{n-1}\right)$$

$$= \frac{1}{\sqrt{5}}\alpha^{n-1}\underbrace{(\alpha+1)}_{\alpha^2} - \frac{1}{\sqrt{5}}\beta^{n-1}\underbrace{(\beta+1)}_{\beta^2}$$

$$= \frac{1}{\sqrt{5}}\left(\alpha^{n+1} - \beta^{n+1}\right). \qquad \text{done.}$$

EXC Use Binet to show $\frac{F_{n+1}}{F_n} \to \alpha$ as $n \to \infty$.

— a more <u>direct</u> proof of Binet uses <u>diagonalization</u>:

$$X_n = \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix}$$ sequence of vectors, sat. matrix equation:

$$X_{n+1} = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}_{\text{call it } A} X_n$$

w. $X_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

<u>Claim</u>: A has eigenvalues $\{\alpha, \beta\}$, eigenvectors $\begin{pmatrix} \alpha \\ 1 \end{pmatrix}$ and $\begin{pmatrix} \beta \\ 1 \end{pmatrix}$.

Char. Poly.: $\det\begin{pmatrix} 1-x & 1 \\ 1 & -x \end{pmatrix} = x^2 - x - 1$

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha+1 \\ \alpha \end{pmatrix} = \begin{pmatrix} \alpha^2 \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} \alpha \\ 1 \end{pmatrix} \text{ etc.}$$

⇝ <u>diagonalize</u> A:

$$A = \begin{pmatrix} \alpha & \beta \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ 1 & 1 \end{pmatrix}^{-1}.$$

$$\Rightarrow A^{n-1} = \begin{pmatrix} \alpha & \beta \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \alpha^{n-1} & 0 \\ 0 & \beta^{n-1} \end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix} 1 & -\beta \\ -1 & \alpha \end{pmatrix}.$$

$$\Rightarrow X_n = A^{n-1}X_1 = \begin{pmatrix} \alpha & \beta \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \alpha^{n-1} & 0 \\ 0 & \beta^{n-1} \end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha & \beta \\ 1 & 1 \end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix} \alpha^{n-1} \\ -\beta^{n-1} \end{pmatrix}$$

$$= \begin{pmatrix} \alpha^n - \beta^n \\ \alpha^{n-1} - \beta^{n-1} \end{pmatrix}\frac{1}{\sqrt{5}} = \begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} \checkmark$$