LECTURE 24
(Mon. MAR. 9, 2020)

§13.4.

X set. A <u>permutation</u> of X is a bijection $f: X \longrightarrow X$.

S_X = set of all permutations of X

[has a notion of "multiplication" = composition].

<u>Note</u>: If X finite, say $|X| = n$, then so is S_X, and

$$|S_X| = \underbrace{n}_{\text{possible } f(1)}\underbrace{(n-1)}_{\text{possible } f(2)} \cdots 2 \cdot \underbrace{1}_{\text{possible } f(n)} = 1 \cdot 2 \cdot 3 \cdots n = n!$$

(<u>convention</u> $0! = 1$)

(may take $X = \{1, 2, \dots, n\}$ for simplicity)

Keep X w. n elements.

<u>Def</u>. For $n \in \mathbb{N}$ and integers $0 \leq r \leq n$, the "<u>binomial coefficient</u>":

$$\binom{n}{r} = \#\{\text{subsets } A \subseteq X \text{ with } |A| = r\}.$$

<u>Exe</u> $\binom{n}{r} = \binom{n}{n-r}$

<u>Hint</u>: $|A| = r \iff |\overline{A}| = n - r$. Consider bijection $A \mapsto \overline{A}$ from $\mathcal{P}(X) \longrightarrow \mathcal{P}(X)$.

<u>Note</u>:

- $\binom{n}{0} = \binom{n}{n} = 1$ ($A = \emptyset$, $A = X$ resp.)

- $\binom{n}{1} = \binom{n}{n-1} = n$

double count: $\{a,b\} = \{b,a\}$.

- $\binom{n}{2} = \binom{n}{n-2} = \frac{1}{2}n(n-1) = T_{n-1}$

<u>Thm.</u> $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{1}{r!}n(n-1)\cdots(n-r+1)$

PROOF. Count the <u>number</u> N of r-<u>tuples</u> (without repetitions) from X in <u>two</u> ways.

$(x_1, x_2, \dots, x_r)$

~ possibilities: n, $n-1$, …, $n-r+1$

$\Rightarrow N = n(n-1)\cdots(n-r+1)$.

On the other hand, first <u>select</u> the elements $\{x_1, \dots, x_r\}$ going into the tuple, then permute them.

$\binom{n}{r}$ ways. $r!$ ways.

$\Rightarrow N = \binom{n}{r} r!$ $\square$

<u>EX.</u> $\binom{n}{3} = \frac{n!}{3!(n-3)!} = \frac{1}{6}n(n-1)(n-2)$.

Pascal's Identity (§13.5): $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$.

PROOF. Let $X = \{1, 2, \ldots, n\}$ and $Y = \{1, 2, \ldots, n-1\}$.

Note: $X = Y \cup \{n\}$ — disjoint.

$$\{A \subseteq X : |A| = r\} = \{A : n \in A\} \cup \{A : n \notin A\}.$$

these are $A = B \cup \{n\}$ where $B \subseteq Y$ has $|B| = r-1$. $\binom{n-1}{r-1}$ of them.

these are subsets $A \subseteq Y$ w. $|A| = r$. $\binom{n-1}{r}$ of them.

$\square$

Exc: Give direct algebraic proof using Thm./factorials.

"Pascal's Triangle":

$$\binom{0}{0}$$

$$\binom{1}{0} \qquad \binom{1}{1}$$

$$\binom{2}{0} + \binom{2}{1} \qquad \binom{2}{2}$$

$$\|$$

$$\binom{3}{0} \qquad \binom{3}{1} \qquad \binom{3}{2} \qquad \binom{3}{3}$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

⋮

Binomial Formula (§13.6): $(a+b)^2 = a^2 + 2ab + b^2$.

$(a+b)^3 = (a+b)(a^2+2ab+b^2) = a^3 + 3a^2b + 3ab^2 + b^3$.

— in general:

Thm. $$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r$$

[ex. $(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$]

PROOF. By induction. Base step $(n=1)$: $(a+b)^1 \overset{?}{=} \sum_{r=0}^{1} \binom{1}{r} a^{1-r} b^r$

(both sides $= a+b$)

Ind. step: Assume Thm. holds for n.
("induction hypothesis").

Then, for exponent $n+1$,

$$(a+b)^{n+1} = (a+b)(a+b)^n = (a+b)\left(\sum_{r=0}^{n}\binom{n}{r} a^{n-r} b^r\right)$$

$$= \sum_{r=0}^{n}\binom{n}{r} a^{n-r+1} b^r \quad + \quad \sum_{r=0}^{n}\binom{n}{r} a^{n-r} b^{r+1}$$

$$= a^{n+1} + \sum_{r=1}^{n}(\cdots) \quad + \quad b^{n+1} + \sum_{r=0}^{n-1}(\cdots)$$

$r=0$ term of 1st sum. $r=n$ term of 2nd sum.

[Change summation variable in 2nd sum; $s = r+1$]

$$= a^{n+1} + \sum_{r=1}^{n} \binom{n}{r} a^{n-r+1} b^r + b^{n+1} + \sum_{s=1}^{n} \binom{n}{s-1} a^{n-s+1} b^s$$

$r=s$

$$= a^{n+1} + b^{n+1} + \sum_{r=1}^{n} \underbrace{\left(\binom{n}{r} + \binom{n}{r-1}\right)}_{\binom{n+1}{r} \text{ by } \underline{\text{Pascal}}.} \cdot a^{n+1-r} b^r$$

$$= \sum_{r=0}^{n+1} \binom{n+1}{r} a^{n+1-r} b^r \qquad \square$$

Cor. $(a=b=1)$: $2^n = \sum_{r=0}^{n} \binom{n}{r}$

exc: Give combinatorial proof.
- using $2^n = |P(X)|$.

$(a=1, b=-1)$: $\sum_{r=0}^{n} \binom{n}{r} (-1)^r = \begin{cases} 1 & n=0 \\ 0 & n>0 \end{cases}$.