LECTURE 3

(Fri. JAN. 10, 2020)

Three subsets $A, B, C \subseteq X$.

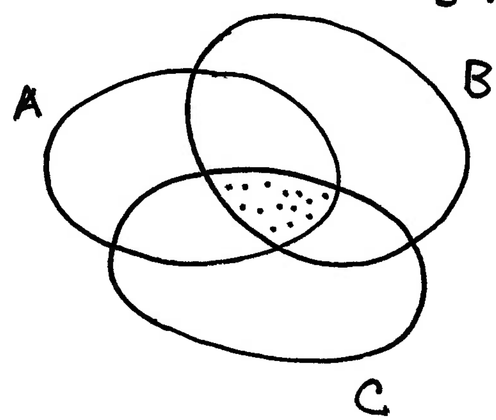
Their <u>union</u>:

$$A \cup B \cup C = \{x : x \in A \text{ or } x \in B \text{ or } x \in C\}$$

— usual interpretation.

Their <u>intersection</u>:

$$A \cap B \cap C = \{x : x \in A \text{ and } x \in B \text{ and } x \in C\}.$$



— Extends to n subsets

$A_1, A_2, \ldots, A_n$

or even a whole (infinite) <u>sequence</u> of subsets:

$A_1, A_2, \ldots$

For instance: By <u>definition</u>,

$$\bigcup_{n=1}^{\infty} A_n = \{x : x \in A_n \text{ for } \underline{\text{some}}\ n \geq 1\}$$

$$\bigcap_{n=1}^{\infty} A_n = \{x : x \in A_n \text{ for } \underline{\text{all}}\ n \geq 1\}.$$

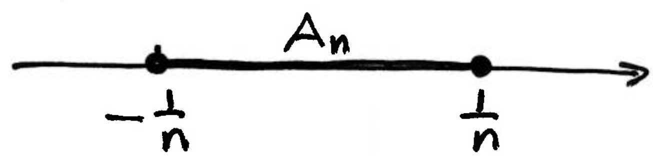
<u>Remark</u>: Analogous to $\sum_{n=1}^{\infty} a_n$ where $a_1, a_2, \ldots$ sequence of numbers.

— the summation index "n" can be called <u>anything</u>. (dummy variable)

$\underline{\text{EX}}$ $(X = \mathbb{R})$ $\quad A_n = \left[-\frac{1}{n}, \frac{1}{n}\right] = \{x \in \mathbb{R} : |x| \leq \frac{1}{n}\}$

$A_1 = [-1, 1]$

$A_2 = \left[-\frac{1}{2}, \frac{1}{2}\right]$ etc.



"decreasing": $\quad A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$

$$\bigcup_{n=1}^{N} A_n = A_1 = [-1, 1], \qquad \bigcap_{n=1}^{N} A_n = A_N = \left[-\frac{1}{N}, \frac{1}{N}\right].$$

For the $\underline{\text{whole}}$ sequence:

$$\bigcap_{n=1}^{\infty} A_n = \{0\}, \qquad \bigcup_{n=1}^{\infty} A_n = A_1 = [-1, 1].$$

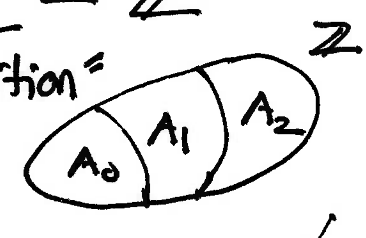
– $\underline{\text{why}}$? Clearly $0 \in A_n$ for all n. (shows $\supseteq$)

$\subseteq$: Suppose $x \in A_n$ for all n, i.e. $|x| \leq \frac{1}{n}$ for all n. Must have $|x| = 0$. Otherwise $|x| > 0$ and $\frac{1}{n} < |x|$ for large enough n. ✓

$\underline{\text{EX}}$ $\quad A_0 = \{3m : m \in \mathbb{Z}\}$ $\qquad A_0 \cup A_1 \cup A_2 = \mathbb{Z}$

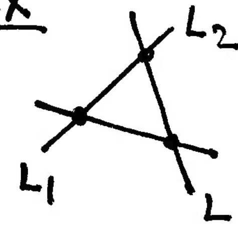
$A_1 = \{3m+1 : m \in \mathbb{Z}\}$ $\qquad$ "partition"

$A_2 = \{3m+2 : m \in \mathbb{Z}\}$



(pairwise disjoint) $\leftarrow$ $\underline{\text{stronger}}$ than: $A_0 \cap A_1 \cap A_2 = \emptyset$

EX



$L_1 \cap L_2 \cap L_3 = \emptyset$
— but not pw. disjoint.

<u>Indexed</u> collections of sets:

Fix an "index set" I.

(above $I = \{1, 2, \dots, n\}$ or $I = \mathbb{N}$ etc.)

Suppose for each $\alpha \in I$ there's an associated subset:

$$A_\alpha \subseteq X.$$

— Now,

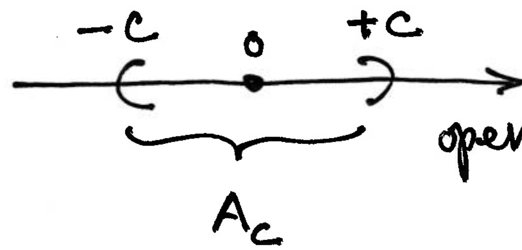
$$\bigcup_{\alpha \in I} A_\alpha = \{x : x \in A_\alpha \text{ for } \cancel{\text{all}}\ \text{some } \alpha \in I\}$$

$$\bigcap_{\alpha \in I} A_\alpha = \{x : x \in A_\alpha \text{ for } \underline{\text{all}}\ \alpha \in I\}.$$

~~Ex~~. ($X = \mathbb{R}$) Take <u>index</u> set $I = \{c \in \mathbb{R} : c > 0\}$ (pos. reals)

For each $c > 0$ consider

$$A_c = (-c, c)$$



open interval

Here:

$$\bigcup_{c>0} A_c = \mathbb{R} \quad \text{and} \quad \bigcap_{c>0} A_c = \{0\}$$

(if $x \neq 0$, then $|x| \geq c$ for some $c > 0$, for ex. $c = |x|$.)

Thm (de MORGAN — extended) For every indexed collection $A_\alpha \subseteq X$, for $\alpha \in I$:

(ex. $A_1, A_2, \ldots$)

(i) $$\overline{\bigcup_{\alpha \in I} A_\alpha} = \bigcap_{\alpha \in I} \overline{A_\alpha}$$

(ii) $$\overline{\bigcap_{\alpha \in I} A_\alpha} = \bigcup_{\alpha \in I} \overline{A_\alpha}$$

— Why? (i): Unwind def., exc. (mimic $A \cup B$ case)

(ii): Apply (i) to $\overline{A_\alpha}$ ✓

- Formal properties: (p. 121)

1) commutative laws: (obvious)
$$\begin{cases} A \cup B = B \cup A \\ A \cap B = B \cap A \end{cases}$$

2) associative laws: (obvious)
$$\begin{cases} A \cup (B \cup C) = (A \cup B) \cup C \\ A \cap (B \cap C) = (A \cap B) \cap C \end{cases}$$

3) distributive laws: (not obvious)
$$\begin{cases} A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \end{cases}$$

compare: For numbers a, b, c,

$a \cdot (b+c) = (a \cdot b) + (a \cdot c)$ etc.