LECTURE 5

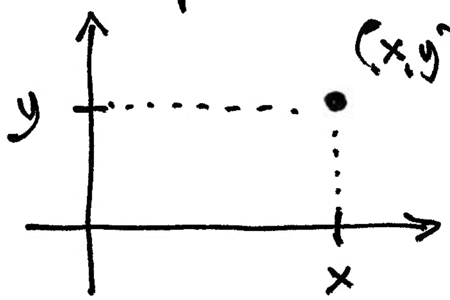
(Wed. JAN. 15, 2020)

Cartesian products: X, Y any two ("universal") sets.

$$X \times Y = \{ (x,y) : x \in X \text{ and } y \in Y \}$$

all ordered pairs: $(x,y) \neq (y,x)$ in general

EX $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ the plane.



Note: $X \times \emptyset = \emptyset$.

EX $\{1,2\} \times \{3,4\} = \{(1,3), (1,4), (2,3), (2,4)\}$.

Theorem. If X, Y are finite, so is $X \times Y$, and

$$|X \times Y| = |X| \cdot |Y|.$$

Why? $X = \{x_1, \dots, x_m\}$, $m = |X|$

$Y = \{y_1, \dots, y_n\}$, $n = |Y|$.

Then,

$$X \times Y = \left.\begin{array}{l} \{ (x_1, y_1), \dots, (x_1, y_n), \\ \quad \vdots \\ \underbrace{(x_m, y_1), \dots, (x_m, y_n)}_{n} \} \end{array}\right] m$$

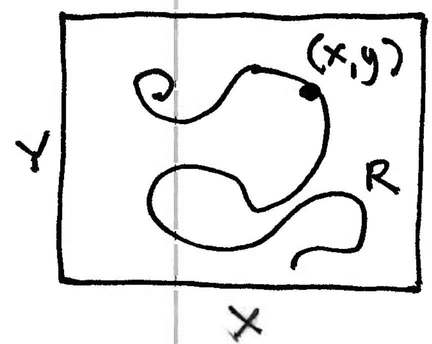
has $m \cdot n$ pairs. ✓

(lexicographic ordering)

LATER: Interested in subsets $R \subseteq X \times Y$
(ex. graphs of functions)

R is also called a "relation" from X to Y.

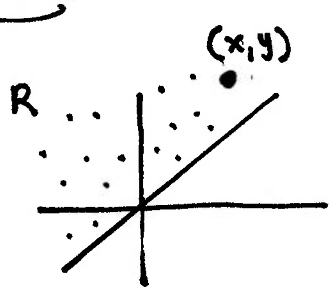
– We'll write xRy when $(x,y) \in R$.



Similarly one def. $X \times Y \times Z$ as the set of triples (x,y,z), and $X_1 \times X_2 \times \cdots \times X_n$ as the set of n-tuples $(x_1, x_2, \ldots, x_n)$ with all $x_i \in X_i$.

EX $\underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n} = \mathbb{R}^n$ Euclidean n-space ("vector space").
↑ can add & scale.

EX $(X = Y = \mathbb{R})$
$x < y$.



EX $(X = Y = \mathbb{Z})$ Let $a, b \in \mathbb{Z}$. We say a "divides" b if $b = ac$ for some $c \in \mathbb{Z}$.

If so, we write $a \mid b$.

This is equivalent to aRb, where

$$R = \{(a,b) \in \mathbb{Z} \times \mathbb{Z} : a \mid b\}$$
(relation)

(note: $1 \mid b$ and $b \mid b$ for any $b \in \mathbb{Z}$)
"trivial divs."
$b = 1 \cdot b = b \cdot 1$

"Mathematical statements", truth value
($P, Q, R, \ldots$)

- true (T)
- false (F)

] not both.

Ex True statements

P: 5 is prime
Q: $\mathbb{Z} \subseteq \mathbb{R}$
R: $\sqrt{2} \in \mathbb{R}$

Ex False statements

A: 6 is odd
B: $\mathbb{R} \subseteq \mathbb{Z}$
C: $\sqrt{2} \in \mathbb{Z}$

Negation: $\sim P$ is the statement "P does not hold"
(ex. $\sim A$: 6 is even)

has truth table:

P	$\sim P$
T	F
F	T

— combining statements.

1) Conjunction:

$P \wedge Q$ "P and Q hold"

2) Disjunction:

$P \vee Q$ "P or Q holds"

possibly both

P	Q	$P \wedge Q$	$P \vee Q$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	F

EX (analogy w. set theory): Fix set X, subsets $A, B \subseteq X$. and an element $x \in X$.

Let P: $x \in A$
Q: $x \in B$.

– Then $\sim P$: $x \in \overline{A}$
$P \wedge Q$: $x \in A \cap B$
$P \vee Q$: $x \in A \cup B$.

(negation $\Leftrightarrow$ complement
conjunction $\Leftrightarrow$ intersection
disjunction $\Leftrightarrow$ union.)

EX a) Fix an $n \in \mathbb{Z}$. Then:

$(n \text{ is odd}) \vee (n+1 \text{ is odd})$ is true.

(if n even, $n = 2m$. Then $n+1 = 2m+1$ odd)

$(n \text{ is odd}) \wedge (n+1 \text{ is odd})$ is false.

b) $(3 \text{ divides } 6) \vee (3 \text{ divides } 8)$ true
$(3 \text{ divides } 6) \wedge (3 \text{ divides } 8)$ false.

false.

Obs: $\sim(P \wedge Q)$ is true precisely when $P \wedge Q$ is false. I.e., when P or Q is false. In other words when $(\sim P) \vee (\sim Q)$ is true.

– say $\sim(P \wedge Q)$ and $(\sim P) \vee (\sim Q)$ are *logically equivalent* statements ("same truth tables") and write

$$\sim(P \wedge Q) \equiv (\sim P) \vee (\sim Q)$$

[compare to de Morgan's law].