LECTURE 6
(Fri. JAN. 17, 2020)

Exc Verify all the other **Boolean** properties (commutative, associative, distributive etc)

Also,

$\sim(\sim P) \equiv P$ — analogous to $\overline{\overline{A}} = A$ for subsets $A \subseteq X$.

Implications: $P \Rightarrow Q$ "If P holds, then Q holds" [or "P holds **only if** Q holds"] "P **implies** Q"

- **Note**: If P is false and Q is (nevertheless) **true**, $P \Rightarrow Q$ is a **true** statement!

Ex: (3 divides 8) $\Rightarrow$ (3 divides 6) is **true**.

- However, **usually** one starts with a **true** P and **deduces** that Q is **true**.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

- **Question**? what's $\sim(P \Rightarrow Q)$
- add column:

$\sim(P \Rightarrow Q)$
F
T
F
F

(can "express" $P \Rightarrow Q$ as the log. equivalent statement

$\sim(P \wedge (\sim Q))$

which uses only $\wedge$ and $\sim$)

- Check (exercise):

$\sim(P \Rightarrow Q) \equiv P \wedge (\sim Q)$

→ Follows formally that

$$P \Rightarrow Q \equiv \sim(P \wedge \sim Q)$$
$$\equiv \sim(\sim Q \wedge P)$$
$$\equiv \sim(\sim Q \wedge \sim\sim P)$$
$$\equiv \underbrace{(\sim Q) \Rightarrow (\sim P)}.$$

– above obs. app. to $\sim Q$ and $\sim P$.

the "contrapositive" of $P \Rightarrow Q$
(if Q does not hold, P doesn't hold)

(compare:
$A \subseteq B$ is equivalent to $\overline{B} \subseteq \overline{A}$)

used for proofs by contradiction:

To show Q is true, assume it's false. From $\sim Q$ deduce a statement which is obviously false.
(P)

Ex $\sqrt{2}$ is irrational.

– For the sake of contradiction, suppose $\sqrt{2} = \frac{a}{b}$ for $a, b \in \mathbb{Z}$ (and $b \neq 0$). We may assume a, b have no common factor > 1.

We know (by taking squares): $2 = \frac{a^2}{b^2}$

— in other words

$$a^2 = 2b^2.$$

part (c)

We see that a^2 is <u>even</u>. By Problem D on HW0, we conclude that a must be even. I.e., $a = 2c$ for some $c \in \mathbb{Z}$.

- Insert this formula:

$$4c^2 = \underbrace{(2c)^2}_{a^2} = 2b^2$$

Can cancel a factor 2 and get $b^2 = 2c^2$, which shows (by repeating the above argument) that b must <u>also</u> be even.

Thus 2 divides both a and b, which <u>contradicts</u> that a, b have no common factor >1.

Have shown, for any two $a, b \in \mathbb{N}$ with <u>no</u> common factor >1 ("coprime")

$$\sqrt{2} = \frac{a}{b} \implies a, b \text{ are both } \underline{\text{even}}.$$

"<u>Contrapositive</u>" is

$$\underbrace{a, b \ \underline{\text{not}} \text{ both even}}_{\text{obviously true when } a,b \text{ are coprime.}} \implies \sqrt{2} \neq \frac{a}{b}.$$

Bi—implications: $P \Longleftrightarrow Q$ "P holds exactly when Q holds" [or "if and only if"].

P	Q	$P \Longleftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

logically equivalent:

$$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

EX i) For any $n \in \mathbb{Z}$,

$$n \text{ even} \Longleftrightarrow n+1 \text{ odd}.$$

ii) For any $x \in \mathbb{R}$,

$$x = 1 \Longrightarrow x^2 = 1$$ true, but $\Longleftarrow$ is false.

What's true:

$$x^2 = 1 \Longleftrightarrow (x=1) \vee (x=-1)$$

Remark: $\sim(P \Leftrightarrow Q) \equiv \sim\big((P \Rightarrow Q) \wedge (Q \Rightarrow P)\big)$

$$\equiv \big(\sim(P \Rightarrow Q)\big) \vee \big(\sim(Q \Rightarrow P)\big)$$

$$\equiv (P \wedge \sim Q) \vee (Q \wedge \sim P)$$

"either P holds and Q doesn't, or vice versa"