LECTURE 7

(Wed. JAN. 22, 2020)

Quantifiers: $\forall$ means "for all"

All $\quad$ $\exists$ means "there exists"

EX: Both true:

- $\forall x \in \mathbb{Z}: \; x^2 \geq 0$
- $\exists x \in \mathbb{Z}: \; x^2 = 1$
- $\forall n \in \mathbb{Z}: \; (n \text{ odd}) \vee (n+1 \text{ odd})$

- Can <u>combine</u> $\forall$ and $\exists$:

EX

- $\forall x \in \mathbb{Z} \; \exists y \in \mathbb{Z}: \; x < y$

  ("for every integer x there is an integer y which is larger"); true, may just take $y = x+1$.

- $\exists x \in \mathbb{Z} \; \forall y \in \mathbb{Z}: \; x < y$

  ("there's an integer x such that every integer y is larger than x"); false, $x-1$ is smaller than x.

EX A an $m \times n$-matrix, $b \in \mathbb{R}^m$.

- To say the <u>linear system</u> $Ax = b$ is <u>consistent</u> for all $b \in \mathbb{R}^m$ (has solutions) amounts to:

$$\forall b \in \mathbb{R}^m \; \exists x \in \mathbb{R}^n: \; Ax = b.$$

– If this is <u>not</u> the case, there's at least <u>some</u> b for which $Ax=b$ has <u>no</u> solutions. That is, $Ax \neq b$ for <u>all</u> $x \in \mathbb{R}^n$.

This amounts to:

$$\exists b \in \mathbb{R}^m \; \forall x \in \mathbb{R}^n : \; Ax \neq b .$$

[note how <u>negation</u> <u>interchanges</u> $\forall$ and $\exists$, and negates the statement $Ax=b$].

— in general:

$$\sim \left(\forall x \in X : \; P(x) \right) \equiv \left(\exists x \in X : \; \sim P(x) \right)$$

same with roles of $\forall$ and $\exists$ reversed.

Example (14.1) "Limits".

A <u>sequence</u> of real numbers: / terms. (order matters)

$$x_1, \; x_2, \; x_3, \; \dots, \; x_n, \; \dots$$

(assigns a real number $x_n \in \mathbb{R}$ to each $n \in \mathbb{N}$)

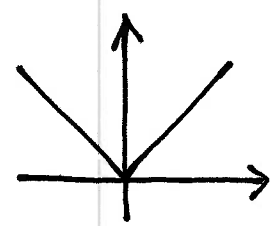
<u>Ex</u>: 1) $x_n = \frac{1}{n}$ gives $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \dots$

2) $y_n = (-1)^n = \begin{cases} 1 & n \text{ even} \\ -1 & n \text{ odd} \end{cases}$ gives $-1, +1, -1, +1, \dots$ "alternates"

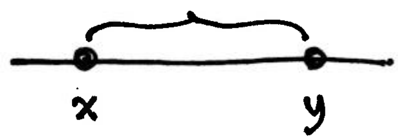
3) $z_n = n^2$ gives $1, 4, 9, 16 \dots$

Recall: the <u>absolute value</u> (of $x \in \mathbb{R}$) is $|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0. \end{cases}$

graph



The <u>distance</u> between $x, y \in \mathbb{R}$ is then $|x-y|$.



A couple of fundamental <u>properties</u>:

i) $|x| = 0 \iff x = 0$.

ii) $|xy| = |x| \cdot |y|$ ("multiplicative")

iii) $|x+y| \leq |x| + |y|$ ("triangle inequality")

iv) $\big|\, |x| - |y| \,\big| \leq |x-y|$.

[ii) is case-by-case. For instance, if $x \geq 0$ and $y < 0$ then $xy \leq 0$ so $|xy| = -xy$. On the other hand $|x| \cdot |y| = x \cdot (-y)$.]

note $|x| = \sqrt{x^2}$.

iii) Enough to check it holds after squaring:

$$
\begin{array}{ccc}
|x+y|^2 & \overset{?}{\leq} & (|x|+|y|)^2 \\
\| & & \| \leftarrow \text{using ii)} \\
(x+y)^2 & & |x|^2+|y|^2+2|xy| \\
\| & & \| \\
x^2+y^2+2xy & & x^2+y^2+2|xy|
\end{array}
$$

indeed $2xy \leq 2|xy|$ (in general $t \leq |t|$)

— argument also shows iii) is equality precisely when $xy \geq 0$. ("same sign")

iv) Follows from iii):

$|x| = |(x-y)+y| \leq |x-y| + |y|$

&

$|y| = |(y-x)+x| \leq |y-x| + |x|$.

Combined: $|x|-|y|$ and $|y|-|x|$ are both $\leq |x-y|$.

— Why "triangle inequality"? Continues to hold for vectors in $\mathbb{R}^2$.

$|x-z| \leq |x-y| + |y-z|$