LECTURE 7
Quantifiers: ∀ means "for all"
   \[ \forall \] means "there exists"
   \[ \exists \]

Ex: Both true:
   - \[ \forall x \in \mathbb{Z}: x^2 \geq 0 \]
   - \[ \exists x \in \mathbb{Z}: x^2 = 1 \]
   - \[ \forall n \in \mathbb{Z}: (n \text{ odd}) \lor (n+1 \text{ odd}) \]

Can combine ∀ and ∃:

Ex:
   - \[ \forall x \in \mathbb{Z} \exists y \in \mathbb{Z}: x < y \]
     ("for every integer x there is an integer y which is larger"); true, may just take \( y = x+1 \).
   - \[ \exists x \in \mathbb{Z} \forall y \in \mathbb{Z}: x < y \]
     ("there's an integer x such that every integer y is larger than x"); false, x-1 is smaller than x.

Ex: A an \( m \times n \)-matrix, \( b \in \mathbb{R}^m \).
   To say the linear system \( Ax = b \) is consistent amounts to:
   \[ \forall b \in \mathbb{R}^m \exists x \in \mathbb{R}^n: Ax = b. \]
   (has solutions)
If this is not the case, there's at least some \( b \) for which \( Ax = b \) has no solutions. That is, \( Ax \neq b \) for all \( x \in \mathbb{R}^n \). This amounts to:

\[
\exists b \in \mathbb{R}^n \forall x \in \mathbb{R}^n: Ax \neq b.
\]

[Note how negation interchanges \( \forall \) and \( \exists \), and negates the statement \( \forall x = b \).]

In general:

\[
\sim (\forall x \in X: p(x)) \equiv (\exists x \in X: \sim p(x))
\]

Same with roles of \( \forall \) and \( \exists \) reversed.
Example (14.1) "Limits."

A sequence of real numbers: / terms.

\[ x_1, x_2, x_3, \ldots, x_n, \ldots \]

(assigns a real number \( x_n \in \mathbb{R} \) to each \( n \in \mathbb{N} \))

Ex: 1) \( x_n = \frac{1}{n} \) gives \( 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots \)

2) \( y_n = (-1)^n \) gives \( -1, +1, -1, +1, \ldots \)

3) \( z_n = n^2 \) gives \( 1, 4, 9, 16, \ldots \)

Recall: The absolute value is \( |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \).

The distance between \( x, y \in \mathbb{R} \) is then \( |x-y| \).

A couple of fundamental properties:

i) \( |x| = 0 \iff x = 0. \)

ii) \( |xy| = |x| \cdot |y| \) ("multiplicative")

iii) \( |x+y| \leq |x| + |y| \) ("triangle inequality")

iv) \( |x|-|y| \leq |x-y| \).
ii) is case-by-case. For instance, if $x > 0$ and $y < 0$ then $xy < 0$ so $|xy| = -xy$. On the other hand, $|x| |y| = x \cdot (-y)$. 

\[ \text{note } |x| = \sqrt{x^2}. \]

iii) Enough to check it holds after squaring:

\[ |x+y|^2 \leq (|x|+|y|)^2 \]

\[ (x+y)^2 \leq |x|^2 + |y|^2 + 2|xy| \]

\[ x^2 + y^2 + 2xy \leq x^2 + y^2 + 2|xy| \]

Indeed $2xy \leq 2|xy|$ (in general $t \leq |t|$)

- Argument also shows iii) is equality precisely when $xy \geq 0$.

\[ \text{"same sign"} \]

iv) Follows from iii):

\[ |x| = |(x-y) + y| \leq |x-y| + |y| \]

\[ |y| = |(y-x) + x| \leq |y-x| + |x| \]

Combined: $|x| - |y|$ and $|y| - |x|$ are both $\leq |x-y|$.

Why "triangle inequality"? Continues to hold for vectors in $\mathbb{R}^2$:

\[ |x-z| \leq |x-y| + |y-z| \]