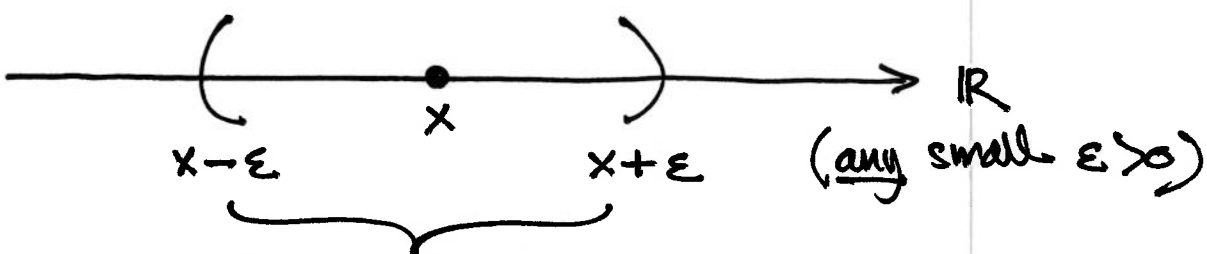
LECTURE 8
(Fri. JAN. 24, 2020)

Back to the <u>sequence</u> (x_n).

- intuitively x_n <u>converges</u> to an $x \in \mathbb{R}$ if x_n gets arbitrarily close to x if we go far enough out in the sequence.



- all but <u>finitely</u> many terms x_n lie in $(x-\varepsilon, x+\varepsilon)$

<u>Def.</u> We say x_n <u>converges</u> to x if for <u>every</u> $\varepsilon>0$ there's an $N \in \mathbb{N}$ s.t. $x_n \in (x-\varepsilon, x+\varepsilon)$ for all $n \geq N$. I.e.,

$$\forall \varepsilon>0\ \exists N\in\mathbb{N}\ \forall n\geq N: \ |x-x_n|<\varepsilon.$$

If so we write

$$x_n \longrightarrow x \quad \text{as} \quad n \to \infty$$

(or just $\lim\limits_{n\to\infty} x_n = x$).

— To say x_n does <u>not</u> converge to x thus amounts to:

$$\exists \varepsilon>0\ \forall N\in\mathbb{N}\ \exists n\geq N: \ |x-x_n|\geq\varepsilon$$

- In words: There's an $\varepsilon>0$ such that <u>infinitely many</u> terms x_n satisfy $|x-x_n|\geq\varepsilon$.

If this holds for <u>all</u> x we say (x_n) <u>diverges</u>.

<u>EX</u> (cont)

1) $x_n=\frac{1}{n}$ <u>converges</u> to 0. Indeed, given an arbitrary $\varepsilon>0$ we can ensure that $|0-x_n|=\frac{1}{n}<\varepsilon$ by taking $n\geq N$ where $N>\frac{1}{\varepsilon}$ is <u>any</u> positive integer.
(say $\varepsilon=0.001$ take any $N>1000$)

2) $y_n=(-1)^n$ <u>diverges</u>. For suppose $y_n\to y$.
Taking $\varepsilon=1$ we find an N s.t. $|y_n-y|<1$ for all $n\geq N$. For such n, noting $\{y_n,y_{n+1}\}=\{\pm1\}$,

$$2=|y_{n+1}-y_n|\leq\underbrace{|y_{n+1}-y|}_{<1}+\underbrace{|y-y_n|}_{<1}<2.$$

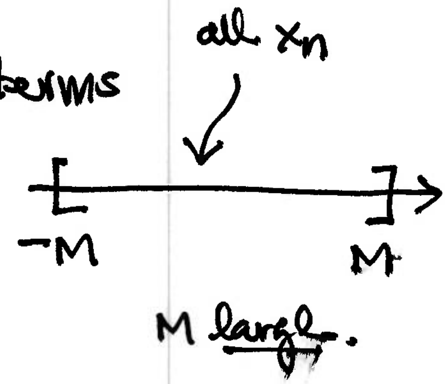
contradiction.

3) $z_n=n^2$ <u>diverges</u>. For suppose $z_n\to z$.
Take $\varepsilon=1$. Get N s.t. $|z_n-z|<1$ as long as $n\geq N$.
Then,

$$\underset{\substack{\| \\ n^2}}{|z_n|}\leq|z_n-z|+|z|<1+|z|$$

contradiction since n^2 <u>unbounded</u>.

~ A sequence (x_n) is "<u>bounded</u>" if all its terms lie in some finite interval. I.e.,



$$\exists M>0 \ \forall n \in \mathbb{N}: \ |x_n| \leqslant M.$$

<u>Rk.</u> Above argument can be adapted to show:

$$(x_n) \text{ convergent} \implies (x_n) \text{ bounded}.$$

Being "<u>unbounded</u>" therefore can be stated as:

$$\forall M>0 \ \exists n \in \mathbb{N}: \ |x_n| > M.$$

("there's always an x_n outside $[-M,M]$ no matter how large M is")

<u>EXC</u>: Show that a sequence can have at most <u>one</u> limit.

I.e., if $x_n \to x$ and $x_n \to y$ (as $n \to \infty$) then $x = y$.

[so notation $\lim\limits_{n\to\infty} x_n$ makes sense]

<u>Hint</u>:

$$\underbrace{|x-y|}_{\text{indep. of } n} \leqslant \underbrace{|x-x_n|}_{\substack{<\frac{\varepsilon}{2} \\ \text{for } n \geqslant N_1}} + \underbrace{|x_n-y|}_{\substack{<\frac{\varepsilon}{2} \\ \text{for } n \geqslant N_2}} < \varepsilon \quad \text{for } N \geqslant \max\{N_1, N_2\} \ \& \ n \geqslant N.$$

Let (x_n) and (y_n) be two convergent sequences.

Say, $x_n \to x$ and $y_n \to y$ as $n \to \infty$.

Theorem (1) $x_n + y_n \to x + y$

(2) $x_n y_n \to xy$

- in other words,

$$\lim_{n\to\infty} x_n + y_n = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n \quad \text{etc.}$$

(interpretation).

PROOF (1): Given any $\varepsilon > 0$ we need to find $N \in \mathbb{N}$ s.t.

$$|(x_n + y_n) - (x + y)| < \varepsilon \quad \text{as long as } n \geq N.$$

Know: There are $A, B \in \mathbb{N}$ s.t.

- $|x_n - x| < \frac{\varepsilon}{2}$ for all $n \geq A$
- $|y_n - y| < \frac{\varepsilon}{2}$ for all $n \geq B$.

Claim $N = \max\{A, B\}$ works. Indeed, for $n \geq N$ we have both bullets above. Therefore, by the triangle inequality:

$$|(x_n + y_n) - (x + y)| = |x_n - x + y_n - y| \leq$$

$$|x_n - x| + |y_n - y| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

PROOF(2): Given any $\varepsilon > 0$ we need to find an $N \in \mathbb{N}$ s.t.

$$|x_n y_n - xy| < \varepsilon \text{ for all } n \geq N.$$

Idea is to write

$$x_n y_n - xy = x_n y_n \underbrace{- x y_n + x y_n}_{\text{cancel}} - xy.$$

$$= (x_n - x) y_n + x(y_n - y).$$

So, $\forall n$:

$$|x_n y_n - xy| \leq |x_n - x| \cdot |y_n| + |x| \cdot |y_n - y|.$$

Know (y_n) is convergent and therefore bounded.
I.e., there's a constant $M > 0$ s.t. $|y_n| \leq M$ for all n.
We may take $M \geq |x|$ if we like.

– Now, we know there are $C, D \in \mathbb{N}$:

- $|x_n - x| < \frac{\varepsilon}{2M}$ for all $n \geq C$
- $|y_n - y| < \frac{\varepsilon}{2M}$ for all $n \geq D$.

Take $N = \max\{C, D\}$. Then for $n \geq N$,

$$|x_n y_n - xy| \leqslant |x_n - x| \cdot |y_n| + |x| \cdot |y_n - y|$$

$$< \frac{\varepsilon}{2M} \cdot M + M \cdot \frac{\varepsilon}{2M} = \varepsilon.$$

$\square$