LECTURE 8
(Fri. JAN. 24, 2020)
Back to the sequence \( (x_n) \).

- Intuitively \( x_n \) converges to an \( x \in \mathbb{R} \) if \( x_n \) gets arbitrarily close to \( x \) if we go far enough out in the sequence.

\[
\begin{array}{c}
\text{IR} \\
\xrightarrow{(\text{any small } \varepsilon > 0)} \\
\text{(all but finitely many)} \\
x_n \text{ lie in } (x-\varepsilon, x+\varepsilon)
\end{array}
\]

**Def.** We say \( x_n \) converges to \( x \) if for every \( \varepsilon > 0 \) there's an \( N \in \mathbb{N} \) s.t. \( x_n \in (x-\varepsilon, x+\varepsilon) \) for all \( n \geq N \).

I.e.,

\[
\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n \geq N: |x - x_n| < \varepsilon.
\]

If so we write

\[
x_n \to x \text{ as } n \to \infty
\]

(or just \( \lim_{n \to \infty} x_n = x \)).

- To say \( x_n \) does not converge to \( x \) thus amounts to:

\[
\exists \varepsilon > 0 \forall N \in \mathbb{N} \exists n \geq N: |x - x_n| \geq \varepsilon
\]
In words: There's an $\varepsilon > 0$ such that infinitely many terms $x_n$ satisfy $|x - x_n| \geq \varepsilon$.

If this holds for all $x$ we say $(x_n)$ diverges.

Ex. (cont.)

1) $x_n = \frac{1}{n}$ converges to 0. Indeed, given an arbitrary $\varepsilon > 0$ we can ensure that $|0 - x_n| = \frac{1}{n} < \varepsilon$ by taking $n \geq N$ where $N > \frac{1}{\varepsilon}$ is any positive integer. (say $\varepsilon = 0.001$ take any $N > 1000$)

2) $y_n = (-1)^n$ diverges. For suppose, $y_n \to y$.

Taking $\varepsilon = 1$ we find an $N$ s.t. $|y_n - y| < 1$ for all $n \geq N$. For such $n$, noting $\{y_n, y_{n+1}\} = \{\pm 1\}$,

$$2 = |y_{n+1} - y_n| \leq |y_{n+1} - y| + |y - y_n| < 2.$$ 

contradiction.

3) $z_n = n^2$ diverges. For suppose, $z_n \to z$.

Take $\varepsilon = 1$. Get $N$ s.t. $|z_n - z| < 1$ as long as $n \geq N$.

Then,

$$|z_n| \leq |z_n - z| + |z| < 1 + |z|$$

$n^2$ contradiction since $n^2$ unbounded.
A sequence \((x_n)\) is "bounded" if all its terms lie in some finite interval. I.e.,

\[\exists M > 0 \forall n \in \mathbb{N}: |x_n| \leq M.\]

Rk. Above argument can be adapted to show:

\((x_n)\) convergent \(\Rightarrow\) \((x_n)\) bounded.

Being "unbounded" therefore can be stated as:

\[\forall M > 0 \exists n \in \mathbb{N}: |x_n| > M.\]

("there's always an \(x_n\) outside \([-M,M]\) no matter how large \(M\) is")

Exc.: Show that a sequence can have at most one limit. I.e., if \(x_n \to x\) and \(x_n \to y\) (as \(n \to \infty\))

then \(x = y\).

[so notation \(\lim_{n \to \infty} x_n\) makes sense.]

\[|x - y| \leq |x - x_n| + |x_n - y| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon\]

for \(n \geq N_1\) for \(n \geq N_2\)
Let \((x_n)\) and \((y_n)\) be two convergent sequences. Say, \(x_n \to x\) and \(y_n \to y\) as \(n \to \infty\).

Theorem

1. \(x_n + y_n \to x + y\)
2. \(x_n y_n \to xy\)

In other words,

\[
\lim_{n \to \infty} x_n + y_n = \lim_{n \to \infty} x_n + \lim_{n \to \infty} y_n \text{ etc.}
\]

(interpretation)

Proof (1): Given any \(\epsilon > 0\) we need to find \(N \in \mathbb{N}\) s.t.

\[
|x_n + y_n - (x + y)| < \epsilon \quad \text{as long as } n \geq N.
\]

Know: There are \(A, B \in \mathbb{N}\) s.t.

- \(|x_n - x| < \frac{\epsilon}{2}\) for all \(n \geq A\)
- \(|y_n - y| < \frac{\epsilon}{2}\) for all \(n \geq B\).

Claim \(N = \max\{A, B\}\) works. Indeed, for \(n \geq N\) we have both bullets above. Therefore, by the triangle inequality:

\[
|x_n + y_n - (x + y)| = |x_n - x + y_n - y| \leq
\]
\[ |x_n - x| + |y_n - y| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \]

**Proof (2):** Given any \( \varepsilon > 0 \) we need to find an \( N \in \mathbb{N} \) such that 
\[ |x_n y_n - xy| < \varepsilon \quad \text{for all } n \geq N. \]

Idea is to write 
\[ x_n y_n - xy = x_n y_n - xy_n + xy_n - xy. \]

\[ = (x_n - x) y_n + x(y_n - y). \]

So, \( \forall n: \)
\[ |x_n y_n - xy| \leq |x_n - x| \cdot |y_n| + |x| \cdot |y_n - y|. \]

Know \( (y_n) \) is convergent and therefore bounded. 
I.e., there's a constant \( M > 0 \) s.t. \( |y_n| \leq M \) for all \( n \). 
We may take \( M \geq |x| \) if we like.

Now, we know there are \( C, D \in \mathbb{N} \):
- \( |x_n - x| < \frac{\varepsilon}{2M} \) for all \( n \geq C \)
- \( |y_n - y| < \frac{\varepsilon}{2M} \) for all \( n \geq D \).

Take \( N = \max\{C, D\} \). Then for \( n \geq N \),
\[ |x_n y_n - xy| \leq |x_n - x| \cdot |y_n| + |x| \cdot |y_n - y| \]

\[ < \frac{\varepsilon}{2M} \cdot M + M \cdot \frac{\varepsilon}{2M} = \varepsilon. \]