LECTURE 9

(Wed. JAN. 29, 2020)

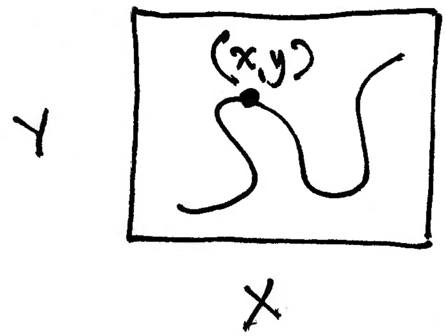
X, Y sets ("universal").

A <u>relation</u> from X to Y is a subset $R \subseteq X \times Y$

all ordered pairs (x,y).

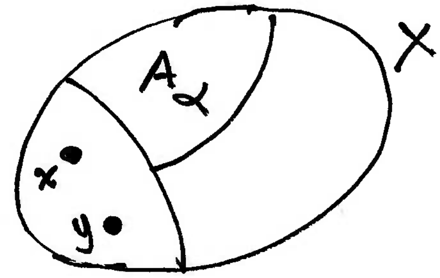
<u>Notation</u>: Write
$(x,y) \in X \times Y$

$$x R y \iff (x,y) \in R.$$



Let $X = Y$.

– Important examples: Suppose X is endowed with a <u>partition</u>, i.e. a collection of non-empty $A_\alpha \subseteq X$ indexed by $\alpha \in I$ s.t. every $x \in X$ lies in exactly one of these subsets:



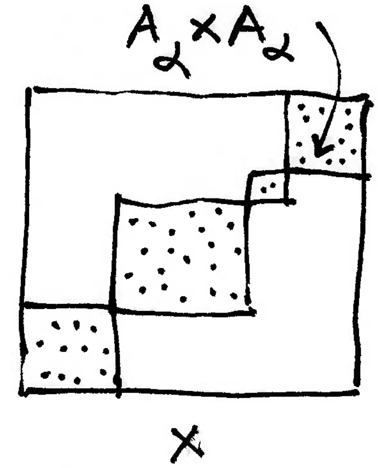
- $X = \bigcup_{\alpha \in I} A_\alpha$
- $A_\alpha \cap A_\beta \neq \emptyset \implies A_\alpha = A_\beta$

Use the partition to define a <u>relation</u> from X to X:

<u>Def</u> $(x, y \in X)$ $x R y$ means x, y lie in the <u>same</u> subset A_α

"roommates"

Here
$$R = \bigcup_{\alpha \in I} A_\alpha \times A_\alpha .$$



- Key properties: (all $x, y, z \in X$)

1) xRx "reflexive"
2) $xRy \iff yRx$ "symmetric".
3) $xRy \wedge yRz \implies xRz$ "transitive".

ANY relation R sat. these properties 1) — 3) is called an equivalence relation.

- we'll show it arises from a partition of X into "equivalence classes".

So suppose $\sim$ is a relation on X s.t.

1) $x \sim x$
2) $x \sim y \iff y \sim x$
3) $x \sim y \wedge y \sim z \implies x \sim z$.

Def. The equivalence class containing x is the subset

$$[x] = \{ y \in X : y \sim x \}.$$

Note 1) amounts to $x \in [x]$ (so non-empty)

☆ GOAL: these subsets $[x] \subseteq X$ (indexed by $I = X$) form a partition of X.

Clearly $X = \bigcup_{x \in X} [x]$. What's missing is the "no overlap" condition.

Claim: $[x] \cap [y] \neq \emptyset \implies [x] = [y]$.
$(x, y \in X)$

→ To see this, preliminary observation: $x, y \in X$,

(*) $[x] = [y] \iff x \sim y$.

Why?

$\implies$: $x \in [x]$ so $x \in [y]$, which means $x \sim y$.

$\impliedby$: If $x \sim y$ then $[x] = [y]$. To see this verify the two inclusions.

- $[x] \subseteq [y]$: Let $z \in [x]$ be any element. This means $z \sim x$. Our hypothesis is $x \sim y$, so by transitivity $z \sim y$, which means $z \in [y]$.
- $[y] \subseteq [x]$: Similar — exc.

✓

– Back to claim: Suppose $[x] \cap [y] \neq \emptyset$.

– pick z.

Then $z \in [x]$ and $z \in [y]$ which means $z \sim x$ and $z \sim y$. By obs. (*): $[z] = [x]$ and $[z] = [y]$.

Consequently $[x] = [z] = [y]$. Done.

Ex Set $M_n(\mathbb{R})$ of all $n \times n$-matrices.

Recall C is invertible if there's an "inverse matrix" C^{-1}:

$$CC^{-1} = C^{-1}C = \underset{\text{identity.}}{I} = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$$

Def. For $A, B \in M_n(\mathbb{R})$ we write $A \sim B$ if there's an invertible C s.t.

$$B = CAC^{-1}$$

[we say A and B are similar].

– This is an equivalence relation:

1) $A \sim A$ since $A = I \cdot A \cdot I^{-1}$

2) $A \sim B$ implies $B \sim A$ $\quad (B = CAC^{-1} \Rightarrow A = C^{-1}B(C^{-1})^{-1})$

3) $A \sim B \wedge B \sim C \Rightarrow A \sim C$.

$\big(B = EAE^{-1}$ and $C = FBF^{-1}$ implies

$C = FEAE^{-1}F^{-1} = (FE)A(FE)^{-1}$ by "socks–shoes" $\big]$