Math 109, Mathematical Reasoning, Midterm Exam Number 1

Monday, January 27th, 2020, 11-11:50am, APM B402A

- *Your Name:* SOLUTIONS
- *ID Number:*
- *Section:*

C01 (4:00 PM) C02 (5:00 PM)

Problem #	Points (out of 10)
1	
2	
3	
4	
5	

Total (out of 50):

Problem 1. Let $X = \{1, 2, 3, \ldots, 10\}$ be the set consisting of all the positive integers less than or equal to 10.

(a) Give its cardinality $|X|$.

(b) How many subsets does X have?

(c) Let $A = \{2, 3, 5, 7\}$. List all the elements of its complement $\bar{A}$ in X.

(a) The cardinality of a finite set is the number of elements. Therefore $|X| = \underline{10}$.

(b) This is the cardinality of the power set $\mathcal{P}(X)$, which is $|\mathcal{P}(X)| = 2^{10} = \underline{1024}$.

(c) $\overline{A}$ consists of those $x \in X$ which are <u>not</u> in A:

$$\boxed{\overline{A} = \{1, 4, 6, 8, 9, 10\}}.$$

Problem 2. Keep the set $X = \{1, 2, 3, \ldots, 10\}$ and the subset $A = \{2, 3, 5, 7\}$ introduced in Problem 1. Furthermore let $B = \{1, 3, 7, 9\}$.

(a) List all elements of their union $A \cup B$ and of their intersection $A \cap B$.

(b) List all elements of the two differences $A - B$ and $B - A$.

(c) Are A and B disjoint subsets?

(a) $A \cup B = \{1, 2, 3, 5, 7, 9\}$ and $A \cap B = \{3, 7\}$.

(b) $A - B = \{2, 5\}$ and $B - A = \{1, 9\}$.

(c) Being <u>disjoint</u> means $A \cap B = \emptyset$.
We found in (a) that $A \cap B = \{3, 7\}$,
so <u>no</u>, they're <u>not</u> disjoint subsets.

Problem 3. Consider the sequence $A_1, A_2, A_3 \ldots$ of intervals in $\mathbb{R}$ defined by $A_n = [0, \frac{1}{n}]$. (Both endpoints 0 and $\frac{1}{n}$ are included.)

(a) Which of the following statements are true, and which are false? Justify your answers.

(1) $A_{2020} \subseteq A_{2019}$

(2) $\frac{1}{2019} \in A_{2020}$

(b) Find their union $\bigcup_{n=1}^{\infty} A_n$ and intersection $\bigcap_{n=1}^{\infty} A_n$.

(c) Do the sets $A_n - A_{n+1}$ form a partition of $(0, 1]$ where $n = 1, 2, 3, \ldots$?

(a) For simplicity we replace 2019 by any $n \geq 1$.

(1) $A_{n+1} \subseteq A_n$ is <u>true</u>: $0 \leq x \leq \frac{1}{n+1}$ implies $0 \leq x \leq \frac{1}{n}$ since $\frac{1}{n+1} < \frac{1}{n}$.

(So we have a <u>decreasing</u> sequence of <u>intervals</u>:

$$\begin{array}{ccccccc} A_1 & \supseteq & A_2 & \supseteq & A_3 & \supseteq & \cdots \\ \parallel & & \parallel & & \parallel & & \\ [0,1] & & [0,\frac{1}{2}] & & [0,\frac{1}{3}] & & \text{etc.} \end{array}$$
)

(2) $\frac{1}{n} \in A_{n+1}$ is <u>false</u>: $0 \leq \frac{1}{n}$ is ok, but the other inequality $\frac{1}{n} \leq \frac{1}{n+1}$ fails.

(b) <u>Union</u>: Since $A_n \subseteq A_1$ for all $n \geq 1$ (as obs. in (1) above) every element of $\bigcup_{n=1}^{\infty} A_n$ lies in A_1.
— and vice versa, obviously.

So,

$$\bigcup_{n=1}^{\infty} A_n = \underline{[0,1]}.$$

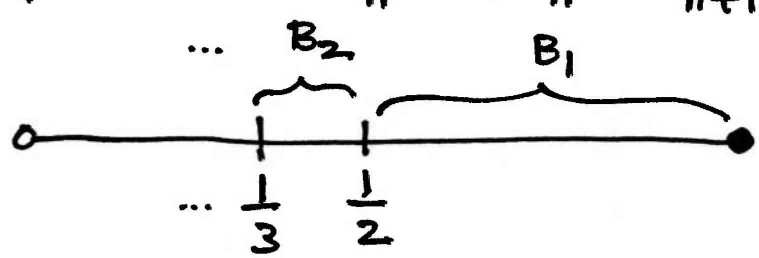
continued →

Intersection: Clearly $0 \in A_n$ for all $n \geq 1$, so at least $\bigcap_{n=1}^{\infty} A_n$ contains 0. Claim 0 is the only element. I.e., that

$$\bigcap_{n=1}^{\infty} A_n = \underline{\{0\}}$$

Why? Suppose x is in the LHS, meaning $0 \leq x \leq \frac{1}{n}$ for all $n \geq 1$. For the sake of contradiction, suppose $x \neq 0$. Then x is positive, and we may pick a large enough N for which $\frac{1}{N} < x$ (just take any integer N larger than $\frac{1}{x}$). This is a contradiction: If $\frac{1}{N} < x$ we cannot have x in $A_N = [0, \frac{1}{N}]$.

(c) Note first that $B_n := A_n - A_{n+1} = \left(\frac{1}{n+1}, \frac{1}{n}\right]$.



$B_1 = (\frac{1}{2}, 1]$

$B_2 = (\frac{1}{3}, \frac{1}{2}]$

$B_3 = (\frac{1}{4}, \frac{1}{3}]$

$\vdots$

The subsets $B_1, B_2, \ldots$ are clearly pairwise disjoint (cf. picture) and

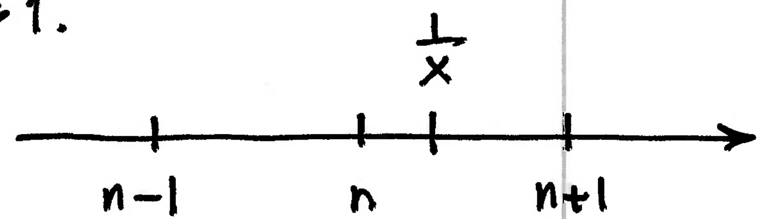
$$\bigcup_{n=1}^{\infty} B_n \stackrel{?}{=} (0, 1]$$

continued

$\subseteq$: Indeed every $B_n = \left(\frac{1}{n+1}, \frac{1}{n}\right] \subseteq (0,1]$.

$\supseteq$: Suppose $0 < x \leqslant 1$. Consider the number $\frac{1}{x} \in \mathbb{R}$ which is $\geqslant 1$.

Pick <u>the</u> $n \in \mathbb{N}$:



$n \leqslant \frac{1}{x} < n+1$

(n is the "<u>floor</u>" of $\frac{1}{x}$)

~ delete digits.

— this amounts to:

$\frac{1}{n+1} < x \leqslant \frac{1}{n}$

In other words $x \in \left(\frac{1}{n+1}, \frac{1}{n}\right] = B_n$ ✓

- CONCLUSION:

– So <u>yes</u>, the collection of intervals $A_n - A_{n+1}$ <u>does</u> form a <u>partition</u> of $(0,1]$.

Problem 4. For each of the mathematical statements below, indicate whether it is true or false. Justify your answers.

TRUE (a) $\forall a \in \mathbb{R} \ \exists b \in \mathbb{N} : a < b$

TRUE (b) $\exists a \in \mathbb{R} \ \forall b \in \mathbb{N} : a < b$

FALSE (c) $\forall n \in \mathbb{Z} : (n > 0) \vee (n < 0)$

FALSE (d) $\exists r \in \mathbb{Q} : (r < 0) \wedge (r > 0)$

FALSE (e) $x \in \mathbb{Q} \Longrightarrow x + \sqrt{2} \in \mathbb{Q}$

(a) Says "for every real number a there's a natural number b which is larger than a"

True: If n is the floor of a, we may take b to be any positive integer $\geq n+1$.

(b) Says "there's a real number a which is smaller than any natural number b".

True: May take $a=0$ (or any real number <1)

(c) Says "every integer n is either positive or negative".

False: $n=0$ is a counterexample.

(d) Says "there's some fraction r which is both negative and positive"

False: No number is both negative and positive.

continued →

(e) Says "if x is a fraction, then so is $x+\sqrt{2}$".

False: $x=0$ is a fraction, but $0+\sqrt{2}=\sqrt{2}$ isn't.

(as shown in class)

Problem 5. Recall that $\mathbb{Z}$ is partitioned into three subsets of numbers:

$$A = \{3n : n \in \mathbb{Z}\} \qquad B = \{3n+1 : n \in \mathbb{Z}\} \qquad C = \{3n+2 : n \in \mathbb{Z}\}.$$

(a) Which of these three subsets contains the number 1000?

(b) Verify the following two implications.

(i) $(x \in B) \wedge (y \in B) \Longrightarrow xy \in B$.

(ii) $(x \in C) \wedge (y \in C) \Longrightarrow xy \in B$.

(a) $1000 = 999 + 1 = 3 \cdot 333 + 1 \in \underline{B}$ (take $n = 333$).

(b)(i) $x, y \in B$ so we may write $x = 3m+1$ and $y = 3n+1$ for suitable $m, n \in \mathbb{Z}$. Then:

$xy = (3m+1)\cdot(3n+1) = 9mn + 3m + 3n + 1 =$

distributive laws.

$3\underbrace{(3mn + m + n)}_{\text{some integer.}} + 1$ is of the form required to be in B.

– This shows $xy \in B$.

(ii) When $x, y \in C$ write $x = 3m+2$ and $y = 3n+2$.

Now,

$xy = (3m+2)(3n+2) = 9mn + 6m + 6n + 4 =$

$3\underbrace{(3mn + 2m + 2n + 1)}_{\text{some integer}} + 1$ is visibly in the subset B.