MATH 109, MATHEMATICAL REASONING, MIDTERM EXAM NUMBER 1

Monday, January 27th, 2020, 11-11:50am, APM B402A

- Your Name: SOLUTIONS
- ID Number:
- Section:
  
  C01 (4:00 PM)  C02 (5:00 PM)

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Total (out of 50):
Problem 1. Let \( X = \{1, 2, 3, \ldots, 10\} \) be the set consisting of all the positive integers less than or equal to 10.

(a) Give its cardinality \(|X|\).

(b) How many subsets does \( X \) have?

(c) Let \( A = \{2, 3, 5, 7\} \). List all the elements of its complement \( \overline{A} \) in \( X \).

\( \overline{A} \) consists of those \( x \in X \) which are not in \( A \):

\[
\overline{A} = \{1, 4, 6, 8, 9, 10\}.
\]
Problem 2. Keep the set $X = \{1, 2, 3, \ldots, 10\}$ and the subset $A = \{2, 3, 5, 7\}$ introduced in Problem 1. Furthermore let $B = \{1, 3, 7, 9\}$.

(a) List all elements of their union $A \cup B$ and of their intersection $A \cap B$.

(b) List all elements of the two differences $A - B$ and $B - A$.

(c) Are $A$ and $B$ disjoint subsets?

\[
\begin{align*}
(a) & \quad A \cup B = \{1, 2, 3, 5, 7, 9, 10\} \quad \text{and} \quad A \cap B = \{3, 7\}.
(b) & \quad A - B = \{2, 5\} \quad \text{and} \quad B - A = \{1, 9\}.
\end{align*}
\]

(c) Being disjoint means $A \cap B = \emptyset$.

We found in (a) that $A \cap B = \{3, 7\}$, so no, they're not disjoint subsets.
Problem 3. Consider the sequence $A_1, A_2, A_3 \ldots$ of intervals in $\mathbb{R}$ defined by $A_n = [0, \frac{1}{n}]$. (Both endpoints 0 and $\frac{1}{n}$ are included.)

(a) Which of the following statements are true, and which are false? Justify your answers.

1. $A_{20\ 000} \subseteq A_{20\ 199}$
2. $\frac{1}{20\ 199} \in A_{20\ 000}$

(b) Find their union $\bigcup_{n=1}^{\infty} A_n$ and intersection $\bigcap_{n=1}^{\infty} A_n$.

(c) Do the sets $A_n - A_{n+1}$ form a partition of $(0, 1]$ where $n = 1, 2, 3, \ldots$?

(a) For simplicity we replace 2019 by any $n \geq 1$.

1. $A_{n+1} \subseteq A_n$ is true: $0 \leq x \leq \frac{1}{n+1}$ implies $0 \leq x \leq \frac{1}{n}$ since $\frac{1}{n+1} < \frac{1}{n}$.

So we have a decreasing sequence of intervals: $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$

$[0,1] \supseteq [0,\frac{1}{2}] \supseteq [0,\frac{1}{3}] \supseteq \cdots$ etc.

2. $\frac{1}{n} \in A_{n+1}$ is false: $0 \leq \frac{1}{n}$ is ok, but the other inequality $\frac{1}{n} \leq \frac{1}{n+1}$ fails.

(b) Union: Since $A_n \subseteq A_1$ for all $n \geq 1$ (as obs. in (1) above) every element of $\bigcap_{n=1}^{\infty} A_n$ lies in $A_1$ — and vice versa, obviously.

So,

$$\bigcup_{n=1}^{\infty} A_n = [0,1].$$

continued
Intersection: Clearly $0 \leq A_n$ for all $n \geq 1$, so at least $\bigcap_{n=1}^{\infty} A_n$ contains $0$. Claim $0$ is the only element, $n=1$ i.e., that

$$\bigcap_{n=1}^{\infty} A_n = \{0\}.$$ 

Why? Suppose $x$ is in the LHS, meaning $0 \leq x \leq \frac{1}{n}$ for all $n \geq 1$. For the sake of contradiction, suppose $x \neq 0$. Then $x$ is positive, and we may pick a large enough $N$ for which $\frac{1}{N} < x$ (just take any integer $N$ larger than $\frac{1}{x}$). This is a contradiction:

If $\frac{1}{N} < x$ we cannot have $x$ in $A_N = [0, \frac{1}{N}]$.

(C) Note, first that $B_n := A_n - A_{n+1} = \left(\frac{1}{n+1}, \frac{1}{n}\right]$.

\[ \cdots \quad B_2 \quad \cdots \quad B_1 \]

\[ \circ \quad \frac{1}{3} \quad \frac{1}{2} \quad \circ \]

The subsets $B_1, B_2, \ldots$ are clearly pairwise disjoint (cf. picture) and

$$\bigcup_{n=1}^{\infty} B_n = (0, 1].$$
\( \leq \): Indeed every \( B_n = \left(\frac{1}{n+1}, \frac{1}{n}\right] \subseteq (0,1] \).

\( \geq \): Suppose \( 0 < x \leq 1 \). Consider the number \( \frac{1}{x} \in \mathbb{R} \) which is \( \geq 1 \).

Pick the \( n \in \mathbb{N} \):

\[
\frac{1}{x} \quad \text{--- this amounts to:} \quad (n \text{ is the "floor" of } \frac{1}{x})
\]

\[
\frac{1}{n+1} < x \leq \frac{1}{n}
\]

--- In other words \( x \leq \left(\frac{1}{n+1}, \frac{1}{n}\right] = B_n \checkmark \)

**CONCLUSION:**

So yes, the collection of intervals \( A_n - A_{n+1} \) does form a partition of \((0,1] \).
Problem 4. For each of the mathematical statements below, indicate whether it is true or false. Justify your answers.

TRUE  
(a) \( \forall a \in \mathbb{R} \exists b \in \mathbb{N}: a < b \)

TRUE  
(b) \( \exists a \in \mathbb{R} \forall b \in \mathbb{N}: a < b \)

FALSE  
(c) \( \forall n \in \mathbb{Z}: (n > 0) \lor (n < 0) \)

FALSE  
(d) \( \exists r \in \mathbb{Q}: (r < 0) \land (r > 0) \)

FALSE  
(e) \( x \in \mathbb{Q} \implies x + \sqrt{2} \in \mathbb{Q} \)

(a) Says "for every real number \( a \), there's a natural number \( b \) which is larger than \( a \)."

- True: If \( n \) is the floor of \( a \), we may take \( b = n + 1 \) to be any positive integer \( \geq n + 1 \).

(b) Says "there's a real number \( a \) which is smaller than any natural number \( b \)."

- True: May take \( a = 0 \) (or any real number \(< 1\)).

(c) Says "every integer \( n \) is either positive or negative."

- False: \( n = 0 \) is a counterexample.

(d) Says "there's some fraction \( r \) which is both negative and positive."

- False: No number is both negative and positive.
(e) Says "if \( x \) is a fraction, then so is \( x + \sqrt{2} \)."

False: \( x = 0 \) is a fraction, but \( 0 + \sqrt{2} = \sqrt{2} \) isn't.

(as shown in class)
Problem 5. Recall that \( Z \) is partitioned into three subsets of numbers:

\[
A = \{ 3n : n \in \mathbb{Z} \} \quad B = \{ 3n + 1 : n \in \mathbb{Z} \} \quad C = \{ 3n + 2 : n \in \mathbb{Z} \}.
\]

(a) Which of these three subsets contains the number 1000?

(b) Verify the following two implications.

(i) \( (x \in B) \land (y \in B) \Rightarrow xy \in B. \)

(ii) \( (x \in C) \land (y \in C) \Rightarrow xy \in B. \)

\[(a) \quad 1000 = 999 + 1 = 3 \cdot 333 + 1 \in B \quad \text{(take } n = 333) . \]

\[(b)(i) \quad x, y \in B \text{ so we may write } x = 3m + 1 \text{ and } y = 3n + 1 \text{ for suitable } m, n \in \mathbb{Z}. \text{ Then:} \]

\[
xy = (3m + 1) \cdot (3n + 1) = 9mn + 3m + 3n + 1 = \]

\[3(3mn + m + n) + 1 \quad \text{is of the form required to be in } B. \]

\[\text{This shows } xy \in B. \]

(ii) When \( x, y \in C \) write \( x = 3m + 2 \) and \( y = 3n + 2 \).

\[
xy = (3m + 2)(3n + 2) = 9mn + 6m + 6n + 4 = \]

\[3(3mn + 2m + 2n + 1) + 1 \quad \text{is visibly in the subset } B. \]