Math 109, Mathematical Reasoning, Midterm Exam Number 2

Monday, February 24th, 2020, 11-11:50am, Peterson Hall 104

- *Your Name*: SOLUTIONS
- *ID Number*:
- *Section*:

C01 (4:00 PM)   C02 (5:00 PM)

Problem #	Points (out of 10)
1	
2	
3	
4	
5	
Total (out of 50):	

Problem 1. Define a relation $\sim$ on the set of real numbers $\mathbb{R}$ by declaring that

$$a \sim b \iff a - b \in \mathbb{Z}.$$

(Here $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ is the set of all integers.)

(a) Show that $\sim$ is an **equivalence** relation on $\mathbb{R}$.

(b) List all x in the range $0 < x < 3$ belonging to the equivalence class of $\sqrt{2}$.

(a). reflexive: $a \sim a$ since $a - a = 0 \in \mathbb{Z}$.

- symmetric: Suppose $a \sim b$, i.e. $a - b \in \mathbb{Z}$. Then $-(a-b) = b - a$ is also an integer, meaning $b \sim a$.
- transitive: Suppose $a \sim b$ and $b \sim c$, i.e. $a - b \in \mathbb{Z}$ and $b - c \in \mathbb{Z}$. Then their sum is also an integer: $(a - b) + (b - c) = a - c \in \mathbb{Z}$

  cancel

  this means $a \sim c$.

(b) The equivalence class of $\sqrt{2}$ is $[\sqrt{2}] = \{a \in \mathbb{R} : a \sim \sqrt{2}\}$

It consists of all numbers of the form $a = \sqrt{2} + t$, for varying $t \in \mathbb{Z}$.

t	$a = \sqrt{2} + t$
-2	$\sqrt{2} - 2$
-1	$\sqrt{2} - 1$
0	$\sqrt{2}$
1	$\sqrt{2} + 1$
2	$\sqrt{2} + 2$

Since $1 < \sqrt{2} < 2$, only the numbers $x =$

$$\boxed{\sqrt{2} - 1, \quad \sqrt{2}, \quad \sqrt{2} + 1}$$

lie in the interval $(0, 3)$

recall, [3] is the set of numbers of the form $3+7t$ as $t \in \mathbb{Z}$ varies.

Problem 2. In this problem [3] denotes the residue class of 3 modulo 7.

(a) Give the integer r in the range $0 \leq r < 7$ satisfying $r \equiv 100 (\text{mod} 7)$.

(b) List all elements of [3] belonging to the open interval $(-20, 20)$.

(c) Give an $x \in [3]$ such that $x^2 \in [3]$ or disprove the existence of such an x.

(a) Observe that: $100 = 14 \cdot 7 + 2 \equiv 2 (\text{mod} 7)$, so $\boxed{r=2}$
("division w. remainder") in the range $0 \leq r < 7$.

(b)

t	$3+7t$
-4	-25
-3	-18
-2	-11
-1	-4
0	3
1	10
2	17
3	24

– only these numbers $3+7t$ lie in the interval $(-20, 20)$:

$$\boxed{-18, -11, -4, 3, 10, 17}$$

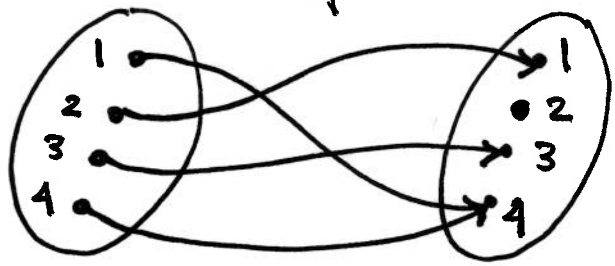
(c) No, such an x does not exist — for the following reason: Saying $x \in [3]$ amounts to $x \equiv 3 (\text{mod} 7)$.
Then, as we know from class, this implies

$$x^2 \equiv 3^2 (\text{mod} 7), \quad \text{i.e.} \quad x^2 \equiv 2 (\text{mod} 7)$$

Cannot have $x^2 \equiv 2 (\text{mod} 7)$ and $x^2 \equiv 3 (\text{mod} 7)$ since [2] and [3] are disjoint. Contradiction.

not injective:
1 and 4
collapse
to a point (4)

f



not surjective:
2 is not in
the range.

Problem 3. Define a function $f : \{1,2,3,4\} \longrightarrow \{1,2,3,4\}$ as follows:

$$f(1) = 4 \qquad f(2) = 1 \qquad f(3) = 3 \qquad f(4) = 4.$$

(a) Is f surjective or injective? (Give a thorough explanation.)

(b) Calculate the value $(f \circ f)(2)$.

(c) List all elements of the two subsets $f(\{1,2\})$ and $f^{-1}(\{3,4\})$.

(a) No. • f is not surjective since there's no x s.t. $f(x)=2$.
and • f is not injective since $f(1)=4=f(4)$, but $1 \neq 4$.

(b) $(f \circ f)(2) = f(f(2)) = f(1) = \boxed{4}$.

(c) Image:

$$f(\{1,2\}) = \{f(1), f(2)\} = \{4,1\} = \boxed{\{1,4\}}.$$

Inverse image:

$$f^{-1}(\{3,4\}) = \{x : f(x)=3 \vee f(x)=4\} = \boxed{\{1,3,4\}}.$$

Solutions?
$x=3$

Solutions?
$x=1$ and $x=4$.

Problem 4.

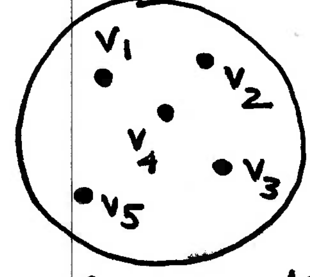
(a) Suppose you are given 11 positive integers. Explain why at least two of them must have the same last digit.

(b) Suppose you are given 5 points in a disk of radius 1. Explain why at least two of them are within a distance $\sqrt{2}$ of each other.

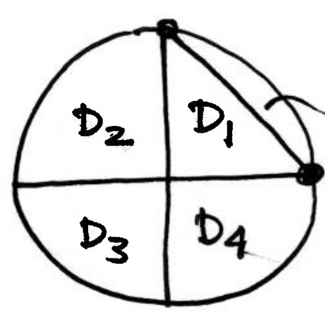
(a) Let $X = \{x_0, x_1, \ldots, x_{10}\}$ be the 11 elements of $\mathbb{N}$.
Consider the function $f: X \longrightarrow \mathbb{Z}_{10}$ given by $f(x) = [x]$.
(taking x to its residue class mod 10). Since $|\mathbb{Z}_{10}| = 10 < |X| = 11$
the "pigeonhole principle" tells us f is not injective.
I.e., there are at least two indices $i \neq j$ in $\{0, 1, \ldots, 10\}$
s.t. $f(x_i) = f(x_j)$. This means $x_i \equiv x_j \pmod{10}$,
in other words they have the same last digit.

(b) Let $A = \{v_1, v_2, v_3, v_4, v_5\}$ be the 5 points in the disk D of radius 1. Divide D into 4 sectors $D = D_1 \cup D_2 \cup D_3 \cup D_4$. By the "pigeonhole principle" at least two points $v_i \neq v_j$ must lie in the same sector, say D_1.
Since D_1 has diameter $\sqrt{2}$,

$d(v_i, v_j) \leq \sqrt{2}$.



(radius = 1)



diam(D_1) =

Problem 5. Let $x_1, x_2, \ldots, x_n, \ldots$ be a sequence of real numbers converging to a number $x \in \mathbb{R}$.

(a) Prove that the sequence is bounded[1].

(b) Give an example of a bounded sequence which is not convergent.

(a) Convergence means $\forall \varepsilon>0\, \exists N\in\mathbb{N}\, \forall n\geqslant N: |x_n - x| < \varepsilon$.
Take $\varepsilon = 1$ and get an $N\in\mathbb{N}$ s.t. $|x_n - x| < 1$ as long as $n \geqslant N$. By the triangle inequality,

$$|x_n| = |(x_n - x) + x| \leqslant |x_n - x| + |x| < 1 + |x|$$

for $n = N, N+1, N+2, \ldots$. Now just take any $B \geqslant 1+|x|$ which is larger than the (finitely many) numbers $|x_1|, |x_2|, \ldots, |x_{N-1}|$. — i.e. larger than their maximum.

(b) Let $c_n = (-1)^n = \begin{cases} 1 & n \text{ even} \\ -1 & n \text{ odd.} \end{cases}$ This bounded: $|c_n| = 1$ for all n. (any $B > 1$ works as a bound)

— Claim (c_n) is not convergent.
For suppose $c_n \to c$. Then,

$$2 = |c_{n+1} - c_n| \leqslant |c_{n+1} - c| + |c - c_n| < 1 + 1 = 2$$

triangle inequality.

for all n large enough (take $\varepsilon = 1$ in def. of convergence)

two consecutive terms are ± 1:
$|1-(-1)| = 2$
$|(-1)-1| = 2.$

Combined, this reads $2 < 2$.
A contradiction, so c_n diverges.

[1] This means there is a constant $B > 0$ such that $|x_n| < B$ for all $n \in \mathbb{N}$.