Math 204A, Fall 2017

Algebraic Number Theory, HW 1

Due Wednesday October 11th 2017 at noon/after class

From [RME]:

- Section 2.4: *5, 6, 7* (all on page 25)

Problem A. Consider the subring $\mathbb{Z}[\sqrt{-2}] = \{a + b\sqrt{-2} : a, b \in \mathbb{Z}\}$ inside $\mathbb{C}$.

(a) Explain why $\mathbb{Q}(\sqrt{-2})$ is the fraction field of $\mathbb{Z}[\sqrt{-2}]$.

(b) Find the units $\mathbb{Z}[\sqrt{-2}]^\times$.

(c) Is $\mathbb{Z}[\sqrt{-2}]$ a principal ideal domain? Prove it or give a counterexample.

(d) Find all integer solutions $x, y \in \mathbb{Z}$ to the equation $y^2 = x^3 - 2$.

Problem B. Let R be a principal ideal domain (PID). Show that R is a unique factorization domain (UFD). (`Note`: You can look up a proof, but you must explain it in your own words.)

Problem C. Consider the ring $\mathbb{Z}[\sqrt{-3}] = \{a + b\sqrt{-3} : a, b \in \mathbb{Z}\}$.

(a) Show that 2 is an irreducible element of $\mathbb{Z}[\sqrt{-3}]$, but not a prime element.

(b) Prove that

$$\mathfrak{p} := (1 + \sqrt{-3}, 1 - \sqrt{-3})$$

is the unique prime ideal of $\mathbb{Z}[\sqrt{-3}]$ containing 2.

(c) Check that $\mathfrak{p}^2 = (2)\mathfrak{p}$.

(d) Conclude that $\mathfrak{p}^n$ is <u>not</u> a principal ideal for any $n > 0$.