From[RME]:

- Section 2.4: 5, 6, 7 (all on page 25)

**Problem A.** Consider the subring $\mathbb{Z}[\sqrt{-2}] = \{a + b\sqrt{-2} : a, b \in \mathbb{Z}\}$ inside $\mathbb{C}$.

(a) Explain why $\mathbb{Q}(\sqrt{-2})$ is the fraction field of $\mathbb{Z}[\sqrt{-2}]$.

(b) Find the units $\mathbb{Z}[\sqrt{-2}]^\times$.

(c) Is $\mathbb{Z}[\sqrt{-2}]$ a principal ideal domain? Prove it or give a counterexample.

(d) Find all integer solutions $x, y \in \mathbb{Z}$ to the equation $y^2 = x^3 - 2$.

**Problem B.** Let $R$ be a principal ideal domain (PID). Show that $R$ is a unique factorization domain (UFD). (**Note:** You can look up a proof, but you must explain it in your own words.)

**Problem C.** Consider the ring $\mathbb{Z}[\sqrt{-3}] = \{a + b\sqrt{-3} : a, b \in \mathbb{Z}\}$.

(a) Show that 2 is an irreducible element of $\mathbb{Z}[\sqrt{-3}]$, but not a prime element.

(b) Prove that

$$p := (1 + \sqrt{-3}, 1 - \sqrt{-3})$$

is the unique prime ideal of $\mathbb{Z}[\sqrt{-3}]$ containing 2.

(c) Check that $p^2 = (2)p$.

(d) Conclude that $p^n$ is **not** a principal ideal for any $n > 0$. 