Math 204B, Winter 2019

Number Theory II, HW 1

Due Wednesday January 16th in class (or by noon).

From Neukirch's book `Algebraic Number Theory`:

- Exercises:

  3 on page 106; 1 and 2 on page 115

Problem A. Let $|\cdot|_1, |\cdot|_2, \dots, |\cdot|_n$ be non-trivial inequivalent absolute values on a field K.

(a) Show that there is an element $a \in K$ with the following properties:

$$|a|_1 > 1, \quad |a|_2 < 1, \quad \dots, \quad |a|_n < 1.$$

(`Hint`. Induction on n. For $n = 2$ use that the open unit ball for $|\cdot|_1$ at 0 is not contained in that of $|\cdot|_2$, and vice versa.)

(b) Let $a_1, \dots, a_n \in K$ be arbitrary elements. Prove that for every $\epsilon > 0$ there exists an $x \in K$ such that

$$|x - a_i|_i < \epsilon \qquad \forall i = 1, 2, \dots, n.$$

(`Hint`. First do the case $a_1 = 1$, $a_2 = \dots = a_n = 0$ by considering $\frac{a^r}{1+a^r}$ for large enough r. In general try $x = a_1x_1 + \dots + a_nx_n$ where x_i is close to 1 relative to $|\cdot|_i$ and close to 0 relative to the others.)

Problem B. Consider the ring of p-adic integers $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$, thought of as the set of compatible residue classes $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \dots)$.

(a) Show that $\mathbb{Z}_p$ is a local domain with maximal ideal $\mathfrak{m}_{\mathbb{Z}_p} = (p) = p\mathbb{Z}_p$.

(b) There are (at least) three natural ways to endow $\mathbb{Z}_p$ with a topology:

   - Taking the ideals $p^n\mathbb{Z}_p$ to be a neighborhood basis at 0;
   - Taking the induced topology from the product $\prod_{n>0} \mathbb{Z}/p^n\mathbb{Z}$;
   - The coarsest topology making the maps $\mathbb{Z}_p \twoheadrightarrow \mathbb{Z}/p^n\mathbb{Z}$ continuous.

   Check that all three give rise to the same topology.